Spontaneous breaking of scale symmetry and nonlinear electrodynamics

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Introduction

- Maxwell's action in vacua is given by $S = \int \sqrt{-g} dx^4 (-\mathcal{F})$ in which $F = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ is the Maxwell invariant.
- Applying the scale transformation $g_{\mu\nu}(x^\mu) \to \Omega^2 g_{\mu\nu}(\Omega x^\mu)$ implies $\sqrt{-g} \to \Omega^4 \sqrt{-g}$ and $F = \frac{1}{4} F_{\alpha\beta} g^{\alpha\lambda} g^{\eta\beta} F_{\lambda\beta} \to \frac{1}{\Omega^4} F$ which leaves the action invariant i.e., $S \to S$.
- Therefore, Maxwell's equations are also scale-invariant and consequently, there is no particle-like solution in Maxwell's linear theory whose field is nonsingular and the self-energy is finite.

Introduction

- One may think of breaking the scale-invariant of the linear theory by introducing a nonlinear version of the theory.
- Nielsen and Olesen in [*H. B. Nielsen, P. Olesen, Nucl. Phys. B 57, 367 (1973)*] proposed a model of nonlinear electrodynamics in the form of $L \sim \sqrt{\mathcal{F}}$ which obviously breaks spontaneously the scale-invariant symmetry.
- In [*P. Gaete, E. I. Guendelman, Phys. Lett. B 640, 201 (2006)*], Maxwell's nonlinear Lagrangian consists of the linear term and an additional scale-invariant breaking term in the form $\mathcal{L}_e = -\frac{1}{4}F - \frac{\mu}{2}\sqrt{-\mathcal{F}}$ has been considered in which μ is the scale-invariant breaking parameter.

Introduction

- This model with the electric field dominance has shown to be very useful in providing a confinement potential in the form of the so-called "Cornell potential" [*E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D 17, 3090 (1978)*] in the context of the quark confinement and in the spectrum of the heavy quark-antiquark system.

- The Cornell potential is given by $V = -\frac{k}{r} + \frac{r}{a^2}$ in which a is a constant of the dimension of length, the first term is the standard Coulomb potential and the second term is a linear potential that provides the quark confinement.

- Let us add that the gauge theory with $L \sim \sqrt{|\mathcal{F}|}$ results in string solution as it was shown in [*A. Aurilia, A. Smailagic, E. Spallucci, Phys. Rev. D 47, 2536 (1993)*] as well as confinement [*E. I. Guendelman, Phys. Lett. B 412, 42 (1997)*].

Scale invariance breaking and nonlinear electrodynamics

We start with the Maxwell action in the flat spacetime which is expressed by

$$S = -\int d^4x\mathcal{F} \tag{1}$$

in which $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell scalar invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ be the electromagnetic field tensor. Variation of the action (1) with respect to the gauge field A_μ yields the standard Maxwell's field equation given by

$$\partial_\mu F^{\mu\nu} = 0. \tag{2}$$

Scale invariance breaking and nonlinear electrodynamics

The action (1) is invariant under the scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$ provided the gauge field is also transformed as $A_\mu \to A'_\mu = \lambda^{-1} A_\mu$. Let us add that λ is a constant dimensionless parameter. Next, we introduce an auxiliary field $\omega\left(x^\mu\right)$ and rewrite the action (1) in the following form

$$S = -\int d^4x \left(\frac{1}{n}\omega^n - \frac{\alpha}{n-1}\omega^{n-1}\left(-\mathcal{F}\right)^{\frac{1}{n}}\right) \tag{3}$$

where α is a constant to be found and $n \geq 2$ is an integer number. Variation of the action with respect to ω implies

$$\omega^{n-1}\left(\omega - \alpha\left(-\mathcal{F}\right)^{\frac{1}{n}}\right) = 0 \tag{4}$$

which yields

$$\omega = \alpha\left(-\mathcal{F}\right)^{\frac{1}{n}}. \tag{5}$$

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Moreover, variation of the action (3) with respect to the Maxwell gauge field yields

$$\partial_\mu \left(\omega^{n-1} \left(-\mathcal{F} \right)^{\frac{1}{n}-1} F^{\mu\nu} \right) = 0 \tag{6}$$

which after considering (5) in (6) we obtain (2). Additionally, considering (5) in the action (3) we get

$$S = \int d^4x \left(\frac{\alpha^n}{n} - \frac{\alpha^n}{n-1} \right) \mathcal{F} \tag{7}$$

which upon setting

$$\alpha = \left(n \left(n-1 \right) \right)^{1/n} \tag{8}$$

it becomes the standard action (1). We also observe that S in (3) remains scale invariant upon

$$\omega \to \lambda^{-4/n} \omega. \tag{9}$$

Scale invariance breaking and nonlinear electrodynamics

Next, we promote the auxiliary field ω, by introducing a second gauge field

$$G_{\alpha\beta\mu\nu} = B_{[\beta\mu\nu,\alpha]} \tag{10}$$

such that

$$\omega = \epsilon^{\alpha\beta\rho\sigma} G_{\alpha\beta\rho\sigma} \tag{11}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the fully antisymmetric Levi-Civita symbol. Note that, $B_{\beta\mu\nu}$ is a three index potential generating the maximal rank tensor $G_{\alpha\beta\mu\nu}$ in four dimensions.

Scale invariance breaking and nonlinear electrodynamics

Furthermore, $G_{\alpha\beta\mu\nu}$ is invariant under the gauge transformation $B_{\beta\mu\nu} \rightarrow B_{\beta\mu\nu} + f_{[\mu\nu,\beta]}$. Upon considering (11), the action (3) becomes

$$S = \int d^4x \left(\frac{1}{n} \left(\epsilon^{\alpha\beta\rho\sigma} G_{\alpha\beta\rho\sigma} \right)^n - \frac{\alpha}{n-1} \left(\epsilon^{\alpha\beta\rho\sigma} G_{\alpha\beta\rho\sigma} \right)^{n-1} \left(-\mathcal{F} \right)^{\frac{1}{n}} \right) \tag{12}$$

such that its variation with respect to A_μ gives Maxwell's equation

$$\partial_\mu \left(F^{\mu\nu} \left(\epsilon^{\alpha\beta\rho\sigma} G_{\alpha\beta\rho\sigma} \right)^{n-1} \left(-\mathcal{F} \right)^{\frac{1}{n}-1} \right) = 0 \tag{13}$$

and with respect to $B_{\xi\mu\nu}$ gives

$$\epsilon^{\alpha\xi\mu\nu} \partial_\alpha \left(\omega^{n-1} - \alpha \omega^{n-2} \left(-\mathcal{F} \right)^{\frac{1}{n}} \right) = 0. \tag{14}$$

Scale invariance breaking and nonlinear electrodynamics

The latter equation implies

$$\omega^{n-1} - \alpha\omega^{n-2}(-\mathcal{F})^{\frac{1}{n}} = M \tag{15}$$

in which M is an integration constant. With $M = 0$ one obtains (5) and therefore Maxwell's equation and the action remain invariant upon assuming (9). However, with $M \neq 0$, first of all (15) itself is not scale invariant and since ω depends on M, Maxwell's equation will no longer be scale invariant. Therefore the scale invariant is broken spontaneously.

To see the mechanism let us consider the specific case with $n = 2$, which has been introduced by Gaete and Guendelman in [*P. Gaete, E. I. Guendelman, Phys. Lett. B 640, 201 (2006)*].

Scale invariance breaking and nonlinear electrodynamics

In this configuration (15) gives

$$\omega = \alpha\sqrt{-\mathcal{F}} + M \tag{16}$$

and Maxwell's equation (13) reduces to

$$\partial_\mu \left(F^{\mu\nu} \left(\alpha + \frac{M}{\sqrt{-\mathcal{F}}} \right) \right) = 0 \tag{17}$$

which clearly due to the existence of $M \neq 0$ (17) is not scale invariant. Furthermore, considering (16) in the action (12) we get (1), while, (17) is considered the field equation of the following action directly

$$S = \int d^4x \left(-\mathcal{F} + 2M\sqrt{-\mathcal{F}} \right) \tag{18}$$

with M the spontaneously braking symmetry parameter. Hence, the nonlinear electrodynamic Lagrangian is found to be

$$\mathcal{L} = -\mathcal{F} + 2M\sqrt{-\mathcal{F}} \tag{19}$$

Scale invariance breaking and nonlinear electrodynamics

The next smallest n is $n = 3$ upon which (15) becomes

$$\omega^2 - \alpha\omega\left(-\mathcal{F}\right)^{\frac{1}{3}} = M \tag{20}$$

that implies

$$\omega = \frac{\alpha\left(-\mathcal{F}\right)^{\frac{1}{3}} \pm \sqrt{\alpha^2\left(-\mathcal{F}\right)^{\frac{2}{3}} + 4C}}{2} \tag{21}$$

which reduces Maxwell's equation to

$$\partial_\mu\left(F^{\mu\nu}\left(1 \pm \sqrt{1 + \frac{2\beta}{\left(-\mathcal{F}\right)^{\frac{2}{3}}}}\right)^2\right) = 0, \tag{22}$$

where $\beta = \frac{2M}{\alpha^2}$. This Maxwell's nonlinear equation corresponds to the following nonlinear electrodynamics model

$$\mathcal{L}_\pm = -\frac{\mathcal{F}}{2}\left[1 + \frac{3\beta}{\mathcal{F}^{2/3}} \pm \left(1 + \frac{2\beta}{\mathcal{F}^{2/3}}\right)^{3/2}\right]. \tag{23}$$

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In the limit $\beta \to 0$ we expect $\mathcal{L} \to -\mathcal{F}$ which implies that the positive branch is acceptable. Therefore we conclude that the nonlinear electrodynamics model corresponding to the spontaneously braking scale invariant with $n = 3$ is the following

$$\mathcal{L} = -\frac{\mathcal{F}}{2}\left[1 + \frac{3\beta}{\mathcal{F}^{2/3}} + \left(1 + \frac{2\beta}{\mathcal{F}^{2/3}}\right)^{3/2}\right] \tag{24}$$

where the $\pm$ index is dropped.

Scale invariance breaking and nonlinear electrodynamics

In the weak field limit one writes

$$\mathcal{L} \to -\sqrt{2}\beta^{3/2} - \frac{3\beta^{1/3}}{2}\mathcal{F}^{1/3} + O\left(\mathcal{F}^{2/3}\right) \quad \text{as } F \to 0 \tag{25}$$

and in the strong field limit we get

$$\mathcal{L} \to -F - 3\beta\mathcal{F}^{1/3} + O\left(\mathcal{F}^{-1/3}\right) \quad \text{as } F \to \infty. \tag{26}$$

For $n > 3$ one can in principle find a nonlinear Lagrangian model, however, the closed form may not be possible. Therefore we concentrate on the case $n = 3$ only.

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Magnetic charged black hole

The action of Einstein's gravity coupled minimally with the nonlinear electrodynamics (19) is given by

$$S = \int \sqrt{-g} dx^4 \left(\frac{\mathcal{R}}{2\kappa^2} + \mathcal{L} \right) \tag{27}$$

where

$$\mathcal{L} = -\frac{\mathcal{F} + \alpha\sqrt{\mathcal{F}}}{4\pi} \tag{28}$$

is the nonlinear electrodynamics Lagrangian, $\kappa^2 = 8\pi G$, R is the Ricci scalar, α is a coupling constant, and $F = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ is Maxwell's invariant.

Magnetic charged black hole

By variating the action with respect to the metric and gauge field we obtain, respectively, Einstein's and Maxwell's field equations given by

$$G_{\mu}^{\nu} = \kappa^2 T_{\mu}^{\nu} \tag{29}$$

and

$$\nabla_{\mu}\left(\mathcal{L}_{\mathcal{F}} F^{\mu\nu}\right) = 0 \tag{30}$$

in which Einstein's tensor is defined to be $G_{\mu}^{\nu} = R_{\mu}^{\nu} - \delta_{\mu}^{\nu} R/2$ and the energy-momentum tensor is expressed by

$$T_{\mu}^{\nu} = \mathcal{L}\delta_{\mu}^{\nu} - \mathcal{L}_{\mathcal{F}} F_{\mu\lambda} F^{\nu\lambda}. \tag{31}$$

Magnetic charged black hole

Next, we consider a static spherically symmetric spacetime with the line element

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{32}$$

in which $f(r)$ is the metric function to be found. Furthermore, the electromagnetic field of the spacetime is provided by a magnetic monopole with the charge $P > 0$ sitting at the center with the following two-form field

$$\mathbf{F} = F_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = P\sin\theta d\theta \wedge d\varphi. \tag{33}$$

Magnetic charged black hole

The only nonzero component of the magnetic field is the radial component given by

$$B_r(r) = \frac{P}{r^2}. \tag{34}$$

Having considered the line element (6), Maxwell's invariant is found to be

$$\mathcal{F} = \frac{1}{2} B_r(r)^2 = \frac{P^2}{2r^4} \tag{35}$$

and clearly both Maxwell's equations and the Bianchi identity

$$\nabla_\mu \left(\tilde{F}^{\mu\nu} \right) = 0 \tag{36}$$

where

$$\tilde{\mathbf{F}} = \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{P}{r^2} dt \wedge dr \tag{37}$$

be the dual electromagnetic field, are satisfied. Coming to the Einstein field equation, all equations reduce to the *tt* component of (3).

Magnetic charged black hole

Knowing that

$$G_t^t = \frac{1}{r^2}\left(f - 1 + rf'\right) \tag{38}$$

and

$$T_t^t = -\frac{\frac{P^2}{2r^4} + \alpha\sqrt{\frac{P^2}{2r^4}}}{4\pi} \tag{39}$$

the main Einstein's equation (the *tt*-component) becomes

$$\frac{1}{r^2}\left(f - 1 + rf'\right) = -\frac{P^2}{r^4} + \alpha\sqrt{\frac{2P^2}{r^4}}. \tag{40}$$

Magnetic charged black hole

Finally, we solve Einstein's field equations to find the metric function given by $(G = 1)$

$$f(r) = 1 - \alpha\sqrt{2}P - \frac{2M}{r} + \frac{P^2}{r^2} \quad (41)$$

where M is an integration constant. We note that with $\alpha = 0$ the Lagrangian and the metric function reduce to Maxwell's linear theory and Reissner-Nordström black hole, respectively. To specify the sign of $\alpha \neq 0$ we apply the energy conditions.

Magnetic charged black hole

To do so, first of all, we calculate the explicit expression of the energy-momentum tensor which is found to be

$$T_{\mu}^{\nu} = diag\left[-\rho, p_r, p_\theta, p_\varphi\right] \tag{42}$$

in which

$$\rho = \frac{1}{8\pi}\left(\frac{P^2}{r^4} + \alpha\frac{\sqrt{2}P}{r^2}\right) \tag{43}$$

is the energy density,

$$p_r = -\rho \tag{44}$$

is the radial pressure and

$$p_\theta, p_\varphi = \frac{P^2}{8\pi r^4} \tag{45}$$

are the angular pressures. The least energy condition to be satisfied by a regular matter is the null energy condition (NEC) which implies

$$\rho + p_i \geq 0. \tag{46}$$

Magnetic charged black hole

This in turn yields

$$\frac{2P^2}{r^4} + \alpha \frac{\sqrt{2}P}{r^2} \geq 0 \tag{47}$$

which is satisfied on $r \geq 0$ only if $\alpha \geq 0$. Hence, we set α to be non-negative, and consequently, not only NEC is satisfied but also all other energy conditions including the weak energy condition $(\rho \geq 0, \rho + p_i \geq 0)$, the strong energy condition $(\rho + \sum_{i=1}^{3} p_i \geq 0)$ and the dominant energy condition $(\rho - |p_i| \geq 0)$ are satisfied.

Magnetic charged black hole

It is important to mention that the spacetime (32) with the metric function (41) is non-asymptotically flat. Hence, the ADM mass is not exactly the constant M in the metric function. Let us introduce $T = \chi t$, $R = \frac{1}{\chi} r$, $M = \chi^3 m$ and $P = \chi^2 Q$, upon which the line element (32) reduces to

$$ds^2 = -f(R)\, dT^2 + \frac{dR^2}{f(R)} + \chi^2 R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{48}$$

where $\chi = \sqrt{1 - \alpha\sqrt{2}P}$ and

$$f(R) = 1 - \frac{2m}{R} + \frac{Q^2}{R^2}. \tag{49}$$

Magnetic charged black hole

Applying the Komar integral [*A. Komar, Phys.Rev. 129, 1873 (1963)*] for the ADM mass given by

$$M_{ADM} = \frac{1}{4\pi} \int_{\partial\Sigma} \sqrt{h} u_\mu n_\nu \left(\nabla^\mu \xi^\nu \right) d^2x, \tag{50}$$

where Σ is the constant t hypersurface with u_μ its timelike normal vector, $\partial\Sigma$ is the timelike boundary of Σ such that n_ν is its normal vector which is spacelike. Therefore $u_\mu u^\mu = -1$ and $n_\mu n^\mu = 1$. Furthermore, h_{ab} is the induced metric on the boundary $\partial\Sigma$ with $h = \det h_{ab}$, and ξ^μ is the normalized timelike killing vector of the spacetime. Considering the line-element (48) one finds $\xi^\nu = \delta^\nu_t$, $u_\mu = -\sqrt{f(R)}\delta^t_\mu$, $n_\nu = \frac{1}{\sqrt{f(R)}}\delta^r_\nu$, $\sqrt{h}d^2x = \chi^2 R^2 \sin\theta d\theta d\varphi$ and consequently

$$u_\mu n_\nu \left(\nabla^\mu \xi^\nu \right) = u_t n_r g^{tt} \nabla_t \xi^r = u_t n_r g^{tt} \Gamma^r_{tt}. \tag{51}$$

Magnetic charged black hole

After knowing the Christoffel symbols $\Gamma^r_{tt} = \frac{1}{2} f(R) f'(R)$ in $R \to \infty$ limit we obtain

$$M_{ADM} = \chi^2 m \tag{52}$$

and in terms of the original integration constant M we get

$$M_{ADM} = \frac{M}{\chi} = \frac{M}{\sqrt{1 - \alpha\sqrt{2}P}}. \tag{53}$$

The spacetime is a singular black hole with a singularity at $r = 0$ and possible horizon(s) at

$$r_{\pm} = \frac{M_{ADM} \pm \sqrt{M^2_{ADM} - P^2}}{\chi}. \tag{54}$$

Magnetic charged black hole

Here M_{ADM} is the ADM mass of the solution and in order to make the spacetime physically acceptable, one should impose $\chi = 1 - \alpha\sqrt{2}P > 0$ upon which if $M_{ADM} > P$ there are two distinct horizons and the black hole contains an event horizon and an inner horizon, if $M_{ADM} = P$ there is only one double (degenerate) horizon and therefore the black hole is extreme, and finally if $M_{ADM} < P$ there is no horizon and the solution is naked singular. Finally, in terms of M_{ADM} the original line element in $\{t, r, \theta, \varphi\}$ coordinates becomes

$$ds^2 = -\left(\chi^2 - \frac{2\chi M_{ADM}}{r} + \frac{P^2}{r^2}\right) dt^2 + \frac{dr^2}{\chi^2 - \frac{2\chi M_{ADM}}{r} + \frac{P^2}{r^2}} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{55}$$

Magnetic charged black hole

In the next section, we shall make a specific TSW based on the bulk spacetime (20). For this reason, we set $\chi = 0$ such that the spacetime falls into the following naked singular structure

$$ds^2 = -\frac{P^2}{r^2}dt^2 + \frac{r^2}{P^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{56}$$

where

$$P = \frac{\sqrt{2}}{2\alpha}. \tag{57}$$

Conclusion

- We introduced a class of nonlinear electrodynamic models which are the result of the spontaneously broken scale-invariant symmetry of the standard Maxwell theory.
- We employed the magnetic dominance version of the nonlinear electrodynamic Lagrangian corresponding to $n = 2$ in the context of spontaneously broken scale-invariant symmetry of the standard Maxwell theory.
- We coupled minimally, this effective Lagrangian to the gravity in the static and spherically symmetric spacetime which was powered by a magnetic monopole.

Conclusion

- We solved the field equations and found a black hole solution in its generic configuration.
- By making the solution finely tuned we get a singular black point spacetime with only one parameter P which is the magnetic charge of the magnetic monopole sitting at the singularity.
- The final spacetime will be used to build the so-called TSW which will be the subject of another talk on Sep. 1 by my colleague Z. Amirabi.