Thin-shell wormhole supported by exotic dust

Zahra Amirabi
Eastern Mediterranean University

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Introduction

- The concept of a thin-shell wormhole (TSW) has been introduced by Matt Visser in [*M. Visser, Phys. Rev. D 39, 3182 (1989)*].
- The idea was to localize and optimize the exotic matter which supports a traversable wormhole [*M. S. Morris, K. S. Thorne, Am. J. Phys., 56, 395 (1988)*], on a thin shell and leave the rest of the bulk spacetime physical (non-exotic).
- In doing so, we pick any physically acceptable bulk spacetime and define a compact timelike hyperplane in the bulk.

Introduction

- Then, we make two identical copies of the outer region of the hyperplane.
- Finally, with the application of the relevant and proper junction conditions, we glue the two incomplete bulks smoothly at their identical boundary which is the compact hyperplane.
- The hyperplane is called the throat of the wormhole (now TSW) which connects two different sides of the throat.

Construction of TSW

The static spherically symmetric bulk spacetime is described by the line element [*Z. Amirabi & S. H. Mazharimousavi, Phys. Scr. 97 095301 (2022)*]

$$ds^2 = -\frac{P^2}{r^2}dt^2 + \frac{r^2}{P^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

where $P = \frac{\sqrt{2}}{2\alpha}$ is the magnetic monopole.

Construction of TSW

- Consider a timelike and spherical hypersurface defined by $\Sigma := \{x | F(r) = r - a(\tau) = 0\}$ in which τ is the proper time on Σ.
- Then we make two identical copies of the outside region of the hypersurface i.e. $M_{\pm} := \{x | F(r) = r - a(\tau) > 0\}$ and glue them on their boundary Σ and make a complete manifold in the form of $M = M_{+} \cup M_{-}$.
- Here, Σ is the interface hypersurface between the two parts and is called the throat.

Construction of TSW

- The line element of the throat(TSW) is given to be

$$ds^2_{TSW} = -d\tau^2 + a^2(\tau)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2}$$

- The two incomplete manifolds M_+ and M_- are mathematically tailored on Σ provided the Israel junction conditions are satisfied [*W. Israel, Nuovo Cimento 44B, 1 (1966)*].
- The first junction condition implies that the induced metric on the TSW should be continuous across the hypersurface Σ.

Construction of TSW

- The form of the line element on the TSW (2) is the result of applying this condition.
- The second junction condition states that the extrinsic curvature tensor or the second fundamental form of the hypersurface is not continuous across the throat such that

$$\left[K_i^j\right] - [K]\,\delta_i^j = -8\pi G S_i^j \tag{3}$$

in which $\left[K_i^j\right] = K_{i(+)}^j - K_{i(-)}^j$ is the discontinuously of the extrinsic curvature tensor and $[K] = \left[K_i^i\right]$ is its trace.

Construction of TSW

- $S_i^j = diag\left[-\sigma, p, p\right]$ is the surface energy-momentum tensor that is required to physically justify the above discontinuity.

Applying (3) implies

$$\sigma = -\frac{\sqrt{f\left(a\right) + \dot{a}^2}}{2\pi a} \tag{4}$$

and

$$p = \frac{\sqrt{f\left(a\right) + \dot{a}^2}}{4\pi}\left(\frac{\ddot{a} + f'\left(a\right)/2}{f\left(a\right) + \dot{a}^2} + \frac{1}{a}\right). \tag{5}$$

Construction of TSW

- At static configuration where $a = a_0$ and $\dot{a} = \ddot{a} = 0$ one finds

$$\sigma_0 = -\frac{\sqrt{f(a_0)}}{2\pi a_0} \tag{6}$$

and

$$p_0 = \frac{\sqrt{f(a_0)}}{8\pi}\left(\frac{f'(a_0)}{f(a_0)} + \frac{2}{a_0}\right). \tag{7}$$

- Considering $f(r) = \frac{P^2}{r^2}$ yields

$$\sigma_0 = -\frac{P}{2\pi a_0^2} \tag{8}$$

and

$$p_0 = 0. \tag{9}$$

Construction of TSW

- The zero pressure/surface-tension for the matter fluid present at the throat, for the static configuration, implies that our TSW is supported by dust, however, since $\sigma_0 < 0$ we shall call it a cloud of exotic dust.
- The total proper mass (in the comoving frame) of the exotic dust can be found as $m_0 = 4\pi a_0^2 \sigma_0 = -2P$.

The dynamic of the throat

Generally, the exotic dust EoS of the fluid present on the shell will no longer remain dust if the static condition of the TSW is changed to dynamic such that the throat's radius becomes time-dependent. In such a configuration one obtains

$$\sigma = -\frac{\sqrt{P^2 + \dot{a}^2 a^2}}{2\pi a^2} \tag{10}$$

and

$$p = \frac{1}{4\pi} \frac{\ddot{a}a + \dot{a}^2}{\sqrt{P^2 + \dot{a}^2 a^2}}. \tag{11}$$

The dynamic of the throat

If we assume that the particles remain non-interacting even in the dynamic phase of the TSW, the dust EoS is applicable i.e., $p = 0$. In such circumstances we get

$$\ddot{a}a + \dot{a}^2 = 0 \tag{12}$$

which clearly implies $\ddot{a} < 0$.

- This in turn means that there is a tendency to collapse the entire throat to a point of radius zero.
- With the proper initial conditions, the first integral of (12) yields

$$\dot{a}a = \dot{a}_0 a_0 \tag{13}$$

and the second integral reveals the radius of the throat in terms of the proper time

$$a^2 = a_0^2 + 2a_0\dot{a}_0\left(\tau - \tau_0\right). \tag{14}$$

The dynamic of the throat

- This is remarkable to observe that in this setting the total energy of the TSW i.e.,

$$m = 4\pi a^2 \sigma \tag{15}$$

is conserved i.e.,

$$m = m_0 = -2\sqrt{P^2 + \dot{a}_0^2 a_0^2} \tag{16}$$

which shows the consistency of the physical configuration.

- Physically Eqs. (13) and (14) imply that with the particles at the TSW's throat non-interacting, if $\dot{a}_0 < 0$ since $\ddot{a} < 0$ the throat will experience an accelerated collapse until $a = 0$.

The dynamic of the throat

- The total finite time needed for the process of collapsing to happen is simply given by Eq. (14) i.e.,

$$\triangle\tau = \frac{a_0}{2\left|\dot{a}_0\right|}. \tag{17}$$

- On the other hand if $\dot{a}_0 > 0$, due to the negative acceleration $\ddot{a} < 0$, the speed of the throat decreases to zero, however, this takes an infinite period of time.
- The reason is that from (12) we get $\dot{a} = \frac{\dot{a}_0 a_0}{a}$ which becomes zero when $a \to \infty$. Consequently , from (14) we find $\triangle\tau \to \infty$.
- In conclusion, the throat either collapses rapidly with increasing acceleration or evaporates with decelerated speed to infinity where the throat practically disappears.

Radial linear perturbation

- We start with the dynamic expression of the surface energy density given in (4) and rewrite it as

$$\dot{a}^2 + \frac{P^2 - \left(2\pi a^2 \sigma\right)^2}{a^2} = 0. \tag{18}$$

- By considering an equilibrium point at $a = a_0$ where $\dot{a} = \ddot{a} = 0$ this equation is satisfied with $\sigma = \sigma_0$.

- To perturb the TSW radially we give an initial velocity $\dot{a}_0 \neq 0$ such that (18) becomes

$$\dot{a}^2 + V(a) = \dot{a}_0^2 \tag{19}$$

in which

$$V(a) = \frac{P^2 - \left(2\pi a^2 \sigma\right)^2}{a^2}. \tag{20}$$

Radial linear perturbation

- Eq. (19) is a one-dimensional equation of motion of a particle of $\frac{mass}{2} = 1$ undergoing a one-dimensional effective potential given by (20) with a total mechanical energy $\varepsilon = \dot{a}_0^2 \neq 0$.
- Furthermore, one can easily show that (4) and (5) are related within the equation

$$\frac{d\sigma}{da} + \frac{2}{a}(p + \sigma) = 0. \tag{21}$$

- The latter equation is nothing but the energy conservation law i.e., $S^{ij}_{;j} = 0$.

Radial linear perturbation

- Considering a fluid with a general variable EoS in the form $p = \psi(a, \sigma)$ we get

$$\frac{d\sigma}{da} = -\frac{2}{a}\left(\psi(a,\sigma) + \sigma\right) \tag{22}$$

$$\frac{d^2\sigma}{da^2} = \frac{2}{a^2}\left[\left(3 + 2\frac{\partial\psi}{\partial\sigma}\right)(\psi + \sigma) - a\frac{\partial\psi}{\partial a}\right]. \tag{23}$$

- Next, we linearize the one-dimensional equation of motion, using the Taylor series expansion of the potential about the equilibrium point i.e.,

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + ... \tag{24}$$

Radial linear perturbation

- Also, considering (22) and (23) together with (8) and (9) in the latter expansion, one obtains $V(a_0) = V'(a_0) = 0$ but

$$V''(a_0) = -\frac{8P\left(Pv^2 + a_0^3 \pi u\right)}{a_0^4} \tag{25}$$

in which

$$u = \left.\frac{\partial \psi(a,\sigma)}{\partial a}\right|_{a=a_0} \tag{26}$$

and

$$v^2 = \left.\frac{\partial \psi(a,\sigma)}{\partial \sigma}\right|_{a=a_0}. \tag{27}$$

Radial linear perturbation

- Note that v is the speed of sound inside the matter fluid present at the throat and therefore $0 \leq v^2 < 1$ where the speed of light is set to one i.e., $c = 1$.
- Considering the above results in (19) we get the linearized equation of motion given by

$$\dot{a}^2 + \frac{1}{2} V''(a_0)(a - a_0)^2 \simeq \dot{a}_0^2. \tag{28}$$

Radial linear perturbation

- It is called linearized since the force field given by $F = -\frac{d}{da}\frac{1}{2}V''(a_0)(a-a_0)^2 = V''(a_0)(a-a_0)$ is linear with respect to $(a-a_0)$. Let us introduce $x = a - a_0$ and therefore $\dot{x} = \dot{a}$ and take the derivative of the of (28) with respect to τ which yields
endframe
$$\ddot{x} + \frac{1}{2}V''(a_0)\,x \simeq 0. \tag{29}$$
- This equation is the well-known one-dimensional simple Harmonic equation if $V''(a_0) > 0$.

Radial linear perturbation

- In this case, the motion of the throat is a Harmonic motion with angular frequency $\omega = \sqrt{\frac{1}{2}V''(a_0)}$.
- Otherwise with $V''(a_0) < 0$ the motion is an exponential motion.
- In conclusion with the Harmonic motion, the TSW is stable against the radial linear perturbation while otherwise the TSW either collapses to $a = 0$ or evaporates to $a \to \infty$, exponentially.

Radial linear perturbation

- Considering the TSW to be stable implies $V''(a_0) > 0$ and therefore
$$-\frac{8P\left(Pv^2 + a_0^3\pi u\right)}{a_0^4} > 0. \tag{30}$$
- This condition is satisfied only if we assume $Pv^2 + a_0^3\pi u < 0$ or
$$u < -\frac{Pv^2}{a_0^3\pi}. \tag{31}$$
- The latter means that $u = \left.\frac{\partial\psi(a,\sigma)}{\partial a}\right|_{a=a_0}$ should be negative and less than $-\frac{Pv^2}{a_0^3\pi}$.

Radial linear perturbation

- In other words, the angular pressure should be a decreasing function with respect to a and the rate of the decrement should be less than $\frac{Pv^2}{a_0^3\pi}$ where v is the speed of sound.
- Therefore having $p = \psi(a, \sigma)$ a decreasing function with respect to a the surface tension becomes an increasing function of a. This condition seems to be physically feasible since increasing the radius of the throat increases the force between the particles on the throat tending to bring it back to the equilibrium radius.

Conclusion

- We applied the method of cut-and-paste introduced by Visser to construct a symmetric TSW whose EoS is identical to a cloud of dust i.e., its surface tension is zero.
- We applied two different methods to investigate its mechanical stability.
- First, we assumed that the EoS of the throat remains unaltered in the dynamic configuration such that the particles stay non-interacting.

  Then we solved the equation of motion of the throat which resulted in instability.

Conclusion

- Second, we relaxed our assumption of being non-interacting the particles on the throat and applied the radial linear perturbation with a variable EoS.
- Our analysis revealed the conditions upon which the TSW becomes stable.
- Static TSW with a dust EoS is possible however its dynamic version is not stable.
- If we allow the particles at the throat to interact with each other in case of a dynamic configuration, most probably the TSW remains stable.