Infrared behavior of two-field cosmological models

Mirela Babalic (& Calin Lazaroiu)

NIPNE, Romania

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- **Multifield cosmological models** (with at least two real scalar fields) are of increasing interest in theoretical physics because:
  - it is more natural to produce multi-field models in fundamental theories of gravity (such as string theory) than to produce one-field models,
  - the 'swampland' conjecture (induced by compatibility with quantum gravity) is very restrictive for a single scalar field.
- In our previous work we initiated a geometric study of the classical dynamics of multifield cosmological models with arbitrary scalar manifold.
  - E. M. Babalic, C. I. Lazaroiu, *The infrared behavior of tame two-field cosmological models*, Nucl. Phys. B 983 (2022), 115929
  - L. Anguelova, E. M. Babalic, C. I. Lazaroiu, *Hidden symmetries of two-field cosmological models*, JHEP 09 (2019) 007
  - L. Anguelova, E. M. Babalic, C. I. Lazaroiu, *Two-field Cosmological α-attractors with Noether Symmetry*, JHEP 04 (2019) 148

Definition

An n-dimensional scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, \mathcal{V})$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected and borderless Riemannian n-dimensional manifold (called scalar manifold)
- $\mathcal{V} \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (to ensure conservation of energy)
2. $\mathcal{V} > 0$ on $\mathcal{M}$ (to avoid certain technical problems)

Each scalar triple defines a cosmological model:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},\mathcal{V}}[g, \varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|} \left[\frac{M^2}{2} R(g) - \frac{1}{2} \mathrm{Tr}_g \varphi^*(\mathcal{G}) - \mathcal{V} \circ \varphi \right]$$

- M is the reduced Planck mass
- g is the metric on the space-time $\mathbb{R}^4$
- $\varphi : \mathbb{R}^4 \longrightarrow \mathcal{M}$
- $\mathcal{V} : \mathcal{M} \longrightarrow \mathbb{R}$
- $\mathrm{Tr}_g \varphi^*(\mathcal{G}) = g^{\mu\nu} \mathcal{G}_{ij} \partial_\mu \varphi^i \partial_\nu \varphi^j \;,\quad \mu, \nu \in \{0, .., 3\} \;,\; i, j \in \{1, ..., n\}$

Take metric g to describe a simply-connected and spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \ \ , \ \ \vec{x} = (x^1, x^2, x^3) \ \ , \ \ a(t) > 0 \ \forall t)$$

and $\varphi = (\varphi^i)_{i=\overline{1,n}}$ to depend only on the cosmological time t, i.e. $\varphi = \varphi(t)$.

When the *Hubble parameter* (defined as $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$) is strictly positive ($H > 0$), then the e.o.m. are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2\mathcal{V}(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\mathcal{V})(\varphi(t)) = 0 \quad , \qquad (1)$$

plus the condition:

$$H(t) = \frac{1}{3M_0}\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2\mathcal{V}(\varphi(t))\right]^{1/2}$$

where $M_0 = M\sqrt{\frac{2}{3}}$ and locally:

$$\nabla_t \dot{\varphi}^i = \ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k \quad , \qquad ||\dot{\varphi}(t)||_{\mathcal{G}}^2 = \mathcal{G}_{ij}\dot{\varphi}^i\dot{\varphi}^j$$

$$\mathrm{grad}_{\mathcal{G}}\mathcal{V} = (\mathrm{grad}_{\mathcal{G}}\mathcal{V})^i\partial_i = \mathcal{G}^{ij}(\partial_j\mathcal{V})\partial_i \ \ , \ \ \partial_i := \frac{\partial}{\partial\varphi^i} \ . \qquad (2)$$

Multifield cosmological models admit a *group of similarities*, which relate cosmological curves for models with the same $\mathcal{M}$ but different M_0, $\mathcal{G}$, $\mathcal{V}$.

Definition

Let $\epsilon > 0$. The *ϵ-scale transform* of a curve $\varphi : I \to \mathcal{M}$ is the curve $\varphi_\epsilon : I_\epsilon \to \mathcal{M}$ defined through:

$$\varphi_\epsilon(t) \stackrel{\text{def.}}{=} \varphi(t/\epsilon) \;\; \forall t \in I_\epsilon \quad , \quad I_\epsilon \stackrel{\text{def.}}{=} \epsilon I = \{\epsilon t | t \in I\} \; .$$

The cosmological equation is invariant under:

- **Scale similarities**:

$$\varphi \to \varphi_\epsilon \quad , \quad \mathcal{V} \to \mathcal{V}_\epsilon \stackrel{\text{def.}}{=} \mathcal{V}/\epsilon^2 \quad (\epsilon > 0) \quad .$$

- **Parameter homotheties**:

$$\mathcal{G} \to \lambda \mathcal{G} \quad , \quad \mathcal{V} \to \lambda \mathcal{V} \quad , \quad M_0 \to \lambda^{1/2} M_0 \quad (\lambda > 0)$$

Definition

The *RG similarity* is the composition of scale similarity at parameter ϵ with parameter homothety at parameter $\lambda = \epsilon^2$:

$$\varphi \to \varphi_\epsilon \quad , \quad M_0 \to \epsilon M_0 \quad , \quad \mathcal{G} \to \epsilon^2 \mathcal{G} \quad , \quad \mathcal{V} \to \mathcal{V}_\epsilon \quad (\epsilon > 0) \quad .$$

A curve $\varphi : I \to \mathcal{M}$ satisfies the cosmological equation of the model $(\mathcal{M}, \mathcal{G}, \mathcal{V})$ iff φ_ϵ satisfies the *ϵ-rescaled cosmological equation*:

$$\nabla_t \dot{\varphi}_\epsilon(t) + \frac{1}{M_0}\left[||\dot{\varphi}_\epsilon(t)||^2_{\mathcal{G}} + 2\mathcal{V}_\epsilon(\varphi_\epsilon(t))\right]^{1/2} \dot{\varphi}_\epsilon(t) + (\mathrm{grad}_{\mathcal{G}} \mathcal{V}_\epsilon)(\varphi_\epsilon(t)) = 0 \quad ,$$

where $\mathcal{V}_\epsilon \stackrel{\text{def.}}{=} \mathcal{V}/\epsilon^2$.

Remark

The IR limit corresponds to slow variation of cosmological curves (low frequency modes of $\varphi(t)$). Equivalently, the limit $\epsilon \to 0$ amounts to $\mathcal{V}_\epsilon \to \infty$.

The IR expansion. When $\epsilon \ll 1$, we expand φ_ϵ in positive powers of ϵ or equivallently expand $\varphi(t)$ in powers of $\frac{1}{\sqrt{2\mathcal{V}}}$.

To first order in the IR expansion, φ is approximated by the solution φ_{IR} of:

$$\boxed{\frac{\mathrm{d}\varphi_{\mathrm{IR}}(t)}{\mathrm{d}t} + (\mathrm{grad}_{\mathcal{G}} V)(\varphi_{\mathrm{IR}}(t)) = 0} \qquad (3)$$

where $V \stackrel{\text{def.}}{=} M_0\sqrt{2\mathcal{V}}$ is the *classical effective scalar potential* of the model.

When $n = 2$ we denote $\mathcal{M} = \Sigma$.

Theorem (Poincaré)

The Weyl equivalence class of any Riemannian metric $\mathcal{G}$ on a borderless connected surface Σ contains a unique complete metric G, called the *uniformizing metric* of $\mathcal{G}$, of constant Gaussian curvature $K = -1$, 0 or $+1$.

- The case $K = -1$ is generic: any metric $\mathcal{G}$ defined on Σ is *hyperbolizable* and its uniformizing metric G is called the *hyperbolization* of $\mathcal{G}$.
- The cases $K = +1$ and $K = 0$ occur only for 7 topologies, as follows:
  - When $K = +1$, the surface Σ must be diffeomorphic with the 2-sphere S^2 or with the real projective plane $\mathbb{RP}^2 \simeq \mathrm{S}^2/\mathbb{Z}_2$.
  - When $K = 0$, the surface Σ must be diffeomorphic with the 2-torus T^2, the Klein bottle $\mathrm{K}^2 = \mathbb{RP}^2 \times \mathbb{RP}^2 \simeq \mathrm{T}^2/\mathbb{Z}_2$, the open annulus A^2, the open Möbius strip $\mathrm{M}^2 \simeq \mathrm{A}^2/\mathbb{Z}_2$ or with the plane $\mathbb{R}^2$.

Cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2\mathcal{V}(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\mathcal{V})(\varphi(t)) = 0 \quad , \qquad (4)$$

Proposition

The **IR behavior** *of the cosmological flow of a two-field model with scalar triple $(\Sigma, \mathcal{G}, \mathcal{V})$ and rescaled Planck mass M_0 is described by the gradient flow of the scalar triple (Σ, G, V), where G is the uniformizing metric of $\mathcal{G}$ and $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\mathcal{V}}$ is the classical effective scalar potential of the model:*

$$\dot{\varphi}_{\mathrm{IR}}(t) + (\mathrm{grad}_G V)(\varphi_{\mathrm{IR}}(t)) = 0 \ . \qquad (5)$$

We will study the IR behaviour near critical points of V and near the ends of Σ.

Types of ends of a topologycal surface

There are 4 hyperbolic types of ends: cusp, plane, horn and funnel ends.

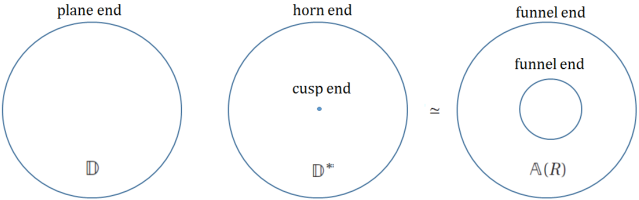


Figure: The elementary hyperbolic surfaces and type of their ends. ($\widehat{\Sigma} = \mathrm{S}^2$)

Reminder

The **ends** of a topological space are, roughly speaking, the connected components of the "ideal boundary" of the space. Each **end** represents a topologically distinct way to move to infinity within the space. Adding a point at each end yields the **end (or Freudenthal) compactification** of the original space, so the set of ends is defined as:

$$\mathrm{Ends}(\Sigma) \stackrel{\text{def.}}{=} \widehat{\Sigma} \setminus \Sigma \iff \widehat{\Sigma} = \Sigma \sqcup \mathrm{Ends}(\Sigma) \ .$$

Definition

A hyperbolizable two-dimensional scalar triple $(\Sigma, \mathcal{G}, \mathcal{V})$ is called *"tame"* if it satisfies the following conditions:

1. Σ is oriented and *topologically finite*. This implies that Σ has finite genus and finite number of ends and that its end compactification $\widehat{\Sigma}$ is a compact smooth surface.
2. The scalar potential $\mathcal{V}$ is *globally well-behaved*, i.e. $\mathcal{V}$ admits a smooth extension $\hat{\mathcal{V}}$ to $\widehat{\Sigma}$. We require that $\hat{\mathcal{V}} > 0$ on $\widehat{\Sigma}$, which means that the limit of $\mathcal{V}$ at each end of Σ is a *strictly* positive number.
3. The extended potential $\hat{\mathcal{V}}$ is a Morse function on $\widehat{\Sigma}$ (in particular, $\mathcal{V}$ is a Morse function on Σ).

A two-field cosmological model with tame scalar triple is called *"tame"*.

To determine the IR behaviour of hyperbolizable "tame" two-field models we will study the gradient flow near interior critical points and near all ends.

Since Σ is topologically finite, then the set of ends $\mathrm{Ends}(\Sigma)$ is finite.

The condition that $\hat{\mathcal{V}}$ is Morse implies that the set of its critical points is finite:

$$\mathrm{Crit}\,\hat{\mathcal{V}} \stackrel{\text{def.}}{=} \{c \in \widehat{\Sigma} | (\mathrm{d}\hat{\mathcal{V}})(c) = 0\} = \mathrm{Crit}\,\widehat{V}$$

The critical points of V coincide with the *interior critical points* of $\widehat{V}$ (and $\hat{\mathcal{V}}$):

$$\mathrm{Crit}\,V = \mathrm{Crit}\mathcal{V} = \Sigma \cap \mathrm{Crit}\,\widehat{V} = \Sigma \cap \mathrm{Crit}\hat{\mathcal{V}} \quad .$$

We have the disjoint union decomposition:

$$\mathrm{Crit}\,\widehat{V} = \mathrm{Crit}\,V \sqcup \mathrm{Crit}_\infty V \quad .$$

where:

$$\mathrm{Crit}_\infty V = \mathrm{Crit}_\infty \mathcal{V} \stackrel{\text{def.}}{=} \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\,\widehat{V} = \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\hat{\mathcal{V}}$$

is the set of *critical ends*.

Any end e of Σ admits an open neighborhood $U_e \subset \widehat{\Sigma}$ diffeomorphic with a disk such that there exist *semigeodesic polar coordinates* $(r, \theta) \in \mathbb{R}_{>0} \times \mathrm{S}^1$ defined on $\dot{U}_e \stackrel{\mathrm{def.}}{=} U_e \setminus \{e\} \subset \Sigma$ in which the metric G has the canonical form:

$$\mathrm{d}s_G^2|_{\dot{U}_e} = \mathrm{d}r^2 + f_e(r)\mathrm{d}\theta^2 \quad ,$$

$$f_e(r) = \begin{cases} \sinh^2(r) & \text{if } e = \text{plane end} \\ \frac{1}{(2\pi)^2} e^{2r} & \text{if } e = \text{horn end} \\ \frac{\ell^2}{(2\pi)^2} \cosh^2(r) & \text{if } e = \text{funnel end of circumference } \ell > 0 \\ \frac{1}{(2\pi)^2} e^{-2r} & \text{if } e = \text{cusp end} \end{cases} .$$

The end e corresponds to $r \to \infty$.

Setting $\omega \stackrel{\mathrm{def.}}{=} \frac{1}{r}$, we have the metric in canonical *polar coordinates* (ω, θ):

$$\mathrm{d}s^2_G|_{\dot{U}_e} = \frac{\mathrm{d}\omega^2}{\omega^4} + f_e(1/\omega)\mathrm{d}\theta^2 \quad ,$$

$$f_e(1/\omega) = \tilde{c}_e e^{\frac{2\varepsilon_e}{\omega}} \left[1 + \mathrm{O}\left(e^{-\frac{2}{\omega}}\right)\right] \quad \text{for} \quad \omega \to 0 \quad ,$$

$$\tilde{c}_e = \begin{cases} \frac{1}{4} & \text{if e} = \text{plane end} \\ \frac{1}{(2\pi)^2} & \text{if e} = \text{horn end} \\ \frac{\ell^2}{(4\pi)^2} & \text{if e} = \text{funnel end of circumference } \ell > 0 \\ \frac{1}{(2\pi)^2} & \text{if e} = \text{cusp end} \end{cases}$$

$$\varepsilon_e = \begin{cases} +1 & \text{if} \quad \text{e} = \text{flaring (i.e. plane, horn or funnel) end} \\ -1 & \text{if} \quad \text{e} = \text{cusp end} \end{cases}$$

$\mathrm{O}\left(e^{-\frac{2}{\omega}}\right) \equiv 0$ when e is a cusp or horn end.

At a critical end e

The Taylor expansion of $\widehat{V}_{\mathrm{e}}$ in principal Cartesian coordinates (x, y) and respectively in polar coordinates (ω, θ):

$$\begin{aligned} \widehat{V}_{\mathrm{e}}(x,y) &= \widehat{V}(e) + \frac{1}{2}\left[\lambda_1(\mathrm{e})x^2 + \lambda_2(\mathrm{e})y^2\right] + \mathrm{O}((x^2+y^2)^{\frac{3}{2}}) \\ \widehat{V}_{\mathrm{e}}(\omega,\theta) &= \widehat{V}(\mathrm{e}) + \frac{1}{2}\omega^2\left[\lambda_1(\mathrm{e})\cos^2\theta + \lambda_2(\mathrm{e})\sin^2\theta\right] + \mathrm{O}(\omega^3)\ , \end{aligned}$$

where $\omega = \sqrt{x^2+y^2}$, $\theta = \arg(x+\mathrm{i}y)$ and the real numbers $\lambda_1(\mathrm{e})$ and $\lambda_2(\mathrm{e})$ are the principal values of the Hessian of $\widehat{V}(\mathrm{e})$.

When λ_1 and λ_2 do not both vanish, it is convenient to define:

Definition

- The *critical modulus* of (Σ, G, V) at the critical end e is the ratio:

$$\beta_{\mathrm{e}} \stackrel{\mathrm{def.}}{=} \frac{\lambda_1(\mathrm{e})}{\lambda_2(\mathrm{e})} \in [-1,1]\setminus\{0\}\ ,$$

  where $\lambda_1(\mathrm{e})$ and $\lambda_2(\mathrm{e})$ are the principal values of (Σ, G, V) at e.
- The *characteristic signs* of (Σ, G, V) at e:

$$\varepsilon_i(\mathrm{e}) \stackrel{\mathrm{def.}}{=} \mathrm{sign}(\lambda_i(\mathrm{e})) \in \{-1,1\}\quad (i=1,2)$$

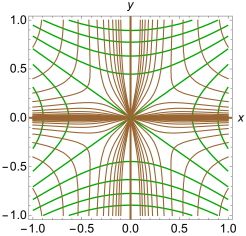
In canonical coordinates (ω, θ), for $\theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, the unoriented gradient flow orbits around the ends are given in the implicit form:

$$\frac{1}{4}[\lambda_1(\mathrm{e}) - \lambda_2(\mathrm{e})]\,\Gamma_2\left(\frac{2\varepsilon_\mathrm{e}}{\omega}\right) = A + \tilde{c}_\mathrm{e}\,[\,\lambda_1(\mathrm{e})\log|\sin\theta| - \lambda_2(\mathrm{e})\log|\cos\theta|\,] \quad , \quad (6)$$

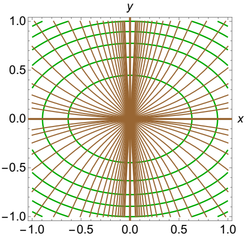
where Γ_2 is the lower incomplete Gamma function of order 2 and A is an integration constant.

We graphically compare the **unoriented gradient flow orbits** to the IR optimal cosmological curves, which are the numerically computed solutions $\varphi(t)$ of the cosmological equation with $\dot{\varphi}(0) = -(\mathrm{grad}_G V)\varphi(0)$. We take $M_0 = 1$ and make certain choices for β.

The IR behaviour near critical plane ends

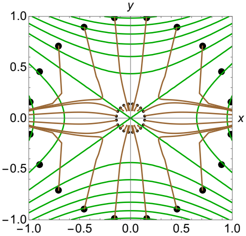


() For $\beta_e = -1/2$

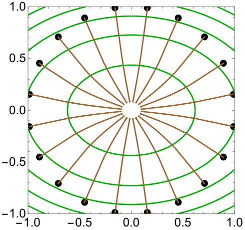


() For $\beta_e = 1/2$

Figure: Gradient flow orbits (brown) over potential lines (green) for critical plane end.

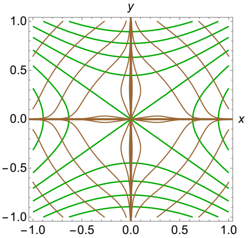


() For $\beta_e = -0.5$

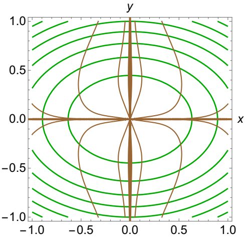


() For $\beta_e = 0.5$

Figure: IR optimal cosmological curves (brown) over potential lines (green). The dots are initial points $\varphi(0)$.

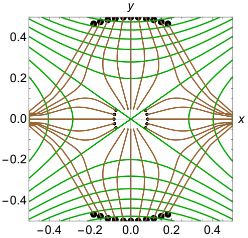


() For $\beta_e = -0.5$.

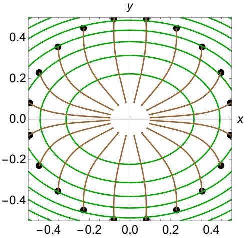


() For $\beta_e = 0.5$.

Figure: Critical horn end.

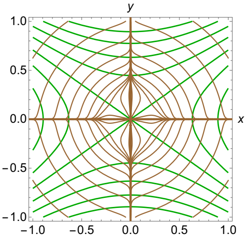


() For $\beta_e = -0.5$.

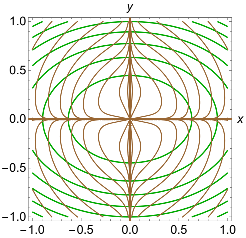


() For $\beta_e = 0.5$.

Figure: Critical horn end. The dots are initial points $\varphi(0)$.

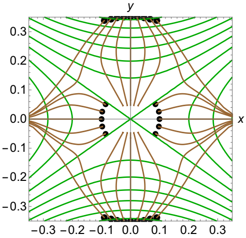


() For $\beta_{\mathrm{e}} = -0.5$.

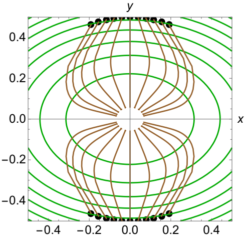


() For $\beta_{\mathrm{e}} = 0.5$.

Figure: Critical funnel end.



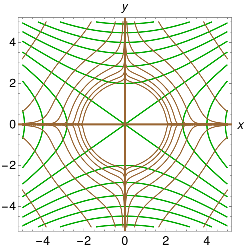
() For $\beta_{\mathrm{e}} = -0.5$.



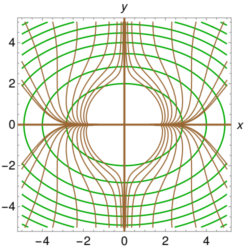
() For $\beta_{\mathrm{e}} = 0.5$.

Figure: Critical funnel end. The dots are initial points $\varphi(0)$.

The IR behaviour near critical cusp ends

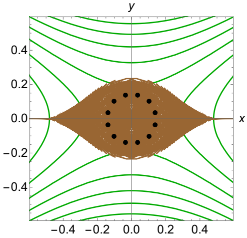


() For $\beta_e = -0.5$.

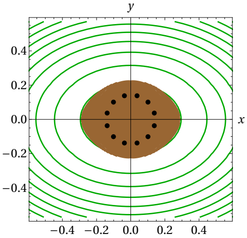


() For $\beta_e = 0.5$.

Figure: Critical cusp end.



() For $\beta_e = -0.5$.



() For $\beta_e = 0.5$.

Figure: Critical cusp end. The dots are initial points $\varphi(0)$.

Note: One must consider higher order corrections in the IR expansion to get better aproximation also for the cusp.

Let c be **an interior critical point** and (x, y) be principal Cartesian canonical coordinates centered at c. We have the metric:

$$\mathrm{d}s_G^2 = \frac{4}{(1-\omega^2)^2}[\mathrm{d}\omega^2 + \omega^2 \mathrm{d}\theta^2]$$

and:

$$V(\omega, \theta) = V(\mathrm{c}) + \frac{1}{2}\omega^2 \left[\lambda_1(\mathrm{c}) \cos^2\theta + \lambda_2(\mathrm{c}) \sin^2\theta\right] + \mathrm{O}(\omega^3) \quad .$$

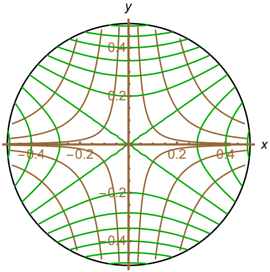
where $\omega \stackrel{\text{def.}}{=} \sqrt{x^2 + y^2}$ and $\theta \stackrel{\text{def.}}{=} \arg(x + \mathrm{i}y)$.

The *critical modulus* β_c and *characteristic signs* $\epsilon_1(\mathrm{c})$ and $\epsilon_2(\mathrm{c})$ of (Σ, G, V) at c are defined through:

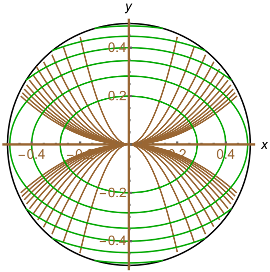
$$\beta_\mathrm{c} \stackrel{\text{def.}}{=} \frac{\lambda_1(\mathrm{c})}{\lambda_2(\mathrm{c})} \in [-1, 1] \setminus \{0\} \quad , \quad \epsilon_i(\mathrm{c}) \stackrel{\text{def.}}{=} \mathrm{sign}(\lambda_i(\mathrm{c})) \;\; (i = 1, 2) \quad .$$

The gradient flow equation gives the general solution: $\omega = C \frac{|\sin(\theta)|^{\frac{\beta_\mathrm{c}}{1-\beta_\mathrm{c}}}}{|\cos(\theta)|^{\frac{1}{1-\beta_\mathrm{c}}}}$, for $\theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, where C is a positive integration constant.

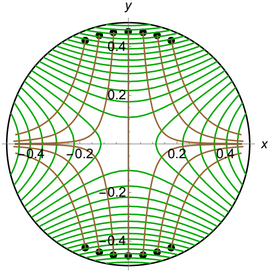
The IR behavior near an interior critical point



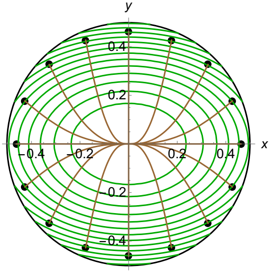
() $\beta_c = -0.5$, saddle point of V.



() $\beta_c = 0.5$, local extremum of V.



() $\beta_c = -0.5$, saddle point of V.
The dots are initial points $\varphi(0)$.



() $\beta_c = 0.5$, local extremum of V.
The dots are initial points $\varphi(0)$.

- We studied the first order IR behavior of "tame" hyperbolizable two-field cosmological models by analyzing the asymptotic form of the gradient flow orbits of the classical effective scalar potential V with respect to the uniformizing metric G near all interior critical points and ends of Σ.
- We showed that the IR behaviour of "tame" hyperbolizable two field cosmological models is characterized by a finite set of parameters associated to their ends and interior critical points.
- Comparing with numerical computations, we found that the first order IR approximation is already quite good for all interior critical points and all ends except for cusps, for which one must consider higher order corrections in the IR expansion in order to obtain a good approximation.
- Our results characterize the IR universality classes of all tame hyperbolizable two-field models in terms of geometric data extracted from the asymptotic behavior of the effective scalar potential and uniformizing metric.

Thank you!