ON A GAUGE INVARIANT VARIABLE FOR SCALAR PERTURBATIONS DURING INFLATION

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Background

The FRWL metric is

$$ds^2 = a^2(\eta)\big\{-d\eta^2 + d\big(x^1\big)^2 + d\big(x^2\big)^2 + d\big(x^3\big)^2\big\},$$

for conformal time $\eta = \int a^{-1}(t)dt$.

The perturbed FRWL metric is

$$\begin{aligned} d\tilde{s}^2 &= a^2(\eta)\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta \\ &= a^2(\eta)\Big\{-(1+2A)d\eta^2 + 2\big(\partial_i B\big)dx^i d\eta \\ &\quad + \big[\big(1 - 2(D + \tfrac{1}{3}(\partial_k\partial_k E))\big)\delta_{ij} \\ &\quad + 2\big(\partial_i\partial_j E\big)\big]dx^i dx^j\Big\}, \end{aligned}$$

for $x^0 = \eta$, and $\mathcal{H} = \dot{a}/a$ below.

Transformation rules of A, B, D, E, $\delta\varphi$

Transformation laws for A, B, D, E, $\delta\varphi$ from the reference frame $O'x'$ to $O''x''$ are

$$\underline{\mathbf{e_0}: A' = A'' - \frac{\partial\xi^0}{\partial\eta} - \xi^0\mathcal{H}},\quad \mathbf{e_{0_0}}: \frac{\partial A'}{\partial\eta} = \frac{\partial A''}{\partial\eta} - \frac{\partial^2\xi^0}{\partial\eta^2} - \frac{\partial\xi^0}{\partial\eta}\mathcal{H} - \xi^0\frac{d\mathcal{H}}{d\eta},$$

$$\underline{\mathbf{e_1}: B' = B'' - \Big(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\Big) + \xi^0},\quad \mathbf{e_{1_0}}: \frac{\partial B'}{\partial\eta} = \frac{\partial B''}{\partial\eta} - \Big(\partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}\Big) + \frac{\partial\xi^0}{\partial\eta},$$

$$\underline{\mathbf{e_2}: D' = D'' + \frac{1}{3}(\partial_i\xi^i) + \xi^0\mathcal{H}},\quad \mathbf{e_{2_0}}: \frac{\partial D'}{\partial\eta} = \frac{\partial D''}{\partial\eta} + \frac{1}{3}(\partial_0\partial_i\xi^i) + \frac{\partial\xi^0}{\partial\eta}\mathcal{H} + \xi^0\frac{d\mathcal{H}}{d\eta},$$

$$\underline{\mathbf{e_3}: \delta\varphi'' = \delta\varphi' - \xi^0\frac{d\overline{\varphi}}{d\eta}},\quad \mathbf{e_{3_0}}: \frac{\partial\delta\varphi'}{\partial\eta} = \frac{\partial\delta\varphi''}{\partial\eta} - \frac{\partial\xi^0}{\partial\eta}\frac{d\overline{\varphi}}{d\eta} - \xi^0\frac{d^2\overline{\varphi}}{d\eta^2},$$

$$\mathbf{e_{3_{00}}}: \frac{\partial^2\delta\varphi'}{\partial\eta^2} = \frac{\partial^2\delta\varphi''}{\partial\eta^2} - \frac{\partial^2\xi^0}{\partial\eta^2}\frac{d\overline{\varphi}}{d\eta} - 2\frac{\partial\xi^0}{\partial\eta}\frac{d^2\overline{\varphi}}{d\eta^2} - \xi^0\frac{d^3\overline{\varphi}}{d\eta^3},$$

$$\underline{\mathbf{e_5}: (\partial_i\partial_i E') = (\partial_i\partial_i E'') - (\partial_i\xi^i)},\quad \mathbf{e_{5_0}}: \frac{\partial}{\partial\eta}(\partial_i\partial_i E') = (\partial_i\partial_i E'') - \frac{\partial}{\partial\eta}(\partial_i\xi^i),$$

$$\mathbf{e_6}: \frac{\partial E'}{\partial\eta} = \frac{\partial E''}{\partial\eta} - \Big(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\Big),\quad \mathbf{e_{6_0}}: \frac{\partial^2 E'}{\partial\eta^2} = \frac{\partial^2 E''}{\partial\eta^2} - \Big(\partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}\Big).$$

Family of invariants

The family of scalar perturbational invariants is

$$\begin{aligned} \mathcal{I}(d_{3_0}, d_{3_{00}}, d_5, d_{5_0}, d_6, d_{6_0}) &= d_0(d_{3_0}, d_{3_{00}}, d_5, d_{5_0}, d_6, d_{6_0})A \\ &+ d_3(d_{3_0}, d_{3_{00}}, d_5, d_{5_0}, d_6, d_{6_0})\delta\varphi \\ &+ d_{3_0}\frac{\partial\delta\varphi}{\partial\eta} + 3d_5\big(D + \frac{1}{3}(\partial_i\partial_i E)\big) \\ &+ 3d_{5_0}\frac{\partial}{\partial\eta}\big(D + \frac{1}{3}(\partial_i\partial_i E)\big) \\ &- d_6\big(B - \frac{\partial E}{\partial\eta}\big) - d_{6_0}\frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big), \end{aligned} \tag{1}$$

for d_{3_0}, $d_{3_{00}}$, d_5, d_{5_0}, d_6, $d_{6_0} \in <R, \mathcal{H}>$

Linearly independent scalar invariants

$$\mathcal{I}^1 = \mathcal{I}(1,0,0,0,0,0) = -\frac{d\overline{\varphi}}{d\eta}A + \Big(\mathcal{H} - \frac{d^2\overline{\varphi}}{d\eta^2}\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\Big)\delta\varphi + \frac{\partial\delta\varphi}{\partial\eta}, \tag{2}$$

$$\mathcal{I}^2 = \mathcal{I}(0,1,0,0,0,0) = \big(\mathcal{H}\frac{d\overline{\varphi}}{d\eta} - 2\frac{d^2\overline{\varphi}}{d\eta^2}\big)A - \frac{d\overline{\varphi}}{d\eta}\frac{\partial A}{\partial\eta} - \Big(\mathcal{H}^2 - \frac{\partial\mathcal{H}}{\partial\eta} - \big(2\mathcal{H}\frac{d^2\overline{\varphi}}{d\eta^2} - \frac{d^3\overline{\varphi}}{d\eta^3}\big)\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\Big)\delta\varphi + \frac{\partial^2\delta\varphi}{\partial\eta^2}, \tag{3}$$

$$\mathcal{I}^3 = \mathcal{I}(0,0,1,0,0,0) = 3D + \big(\partial_i\partial_i E\big) + 3\mathcal{H}\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\delta\varphi, \tag{4}$$

$$\mathcal{I}^4 = \mathcal{I}(0,0,0,1,0,0) = 3\mathcal{H}A + \frac{\partial}{\partial\eta}\Big(3D + \big(\partial_i\partial_i E\big)\Big) - 3\big(\mathcal{H}^2 - \frac{\partial\mathcal{H}}{\partial\eta}\big)\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\delta\varphi, \tag{5}$$

$$\mathcal{I}^5 = \mathcal{I}(0,0,0,0,1,0) = -\big(B - \frac{\partial E}{\partial\eta}\big) - \big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\delta\varphi, \tag{6}$$

$$\mathcal{I}^6 = \mathcal{I}(0,0,0,0,0,1) = -A - \frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big) + \mathcal{H}\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\delta\varphi. \tag{7}$$

$$\begin{aligned} &\widetilde{\mathcal{I}}^1 = -\big(\frac{d\overline{\varphi}}{d\eta}\big)^{-1}\mathcal{I}^1 = A + \mathcal{F}^1(\delta\varphi), \quad \widetilde{\mathcal{I}}^3 = \frac{1}{3}\mathcal{I}^3 = \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + \mathcal{F}^3(\delta\varphi), \quad \widetilde{\mathcal{I}}^5 = -\mathcal{I}^5 = \Big(B - \frac{\partial E}{\partial\eta}\Big) + \mathcal{F}^5(\delta\varphi), \\ &\widetilde{\mathcal{I}}^2 = \partial\widetilde{\mathcal{I}}^1/\partial\eta, \qquad \widetilde{\mathcal{I}}^4 = \partial\widetilde{\mathcal{I}}^3/\partial\eta, \qquad \widetilde{\mathcal{I}}^6 = \partial\widetilde{\mathcal{I}}^5/\partial\eta. \end{aligned} \tag{8}$$

Results

First Bardeen's potential: $$\Phi = A + \mathcal{H}B + \frac{\partial B}{\partial\eta} - \mathcal{H}\frac{\partial E}{\partial\eta} - \frac{\partial^2 E}{\partial\eta^2} = -\mathcal{H}\mathcal{J}^5 - \mathcal{J}^6 = \widetilde{\mathcal{J}}^1 + \mathcal{H}\widetilde{\mathcal{J}}^5 + \widetilde{\mathcal{J}}^6, \tag{9}$$

Second Bardeen's potential: $$\Psi = D + \frac{1}{3}\big(\partial_i\partial_i E\big) - \mathcal{H}B + \mathcal{H}\frac{\partial E}{\partial\eta} = \frac{1}{3}\mathcal{J}^3 + \mathcal{H}\mathcal{J}^5 = \widetilde{\mathcal{J}}^3 - \mathcal{H}\widetilde{\mathcal{J}}^5, \tag{10}$$

Invariant scalar field perturbation: $$\delta\varphi_{INV} = \frac{d\overline{\varphi}}{d\eta}B - \frac{d\overline{\varphi}}{d\eta}\frac{\partial E}{\partial\eta} + \delta\varphi = -\frac{d\overline{\varphi}}{d\eta}\mathcal{J}^5 = \frac{d\overline{\varphi}}{d\eta}\widetilde{\mathcal{J}}^5, \tag{11}$$

Mukhanov-Sasaki variable: $$v = a\mathcal{H}^{-1}\frac{d\overline{\varphi}}{d\eta}\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + a\delta\varphi = \frac{1}{3}a\mathcal{H}^{-1}\frac{d\overline{\varphi}}{d\eta}\mathcal{J}^3 = a\mathcal{H}^{-1}\frac{d\overline{\varphi}}{d\eta}\widetilde{\mathcal{J}}^3. \tag{12}$$

In the base A, $\frac{\partial A}{\partial\eta}$, $B - \frac{\partial E}{\partial\eta}$, $\frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big)$, $3D + (\partial_i\partial_i E)$, $\frac{\partial}{\partial\eta}\big(3D + (\partial_i\partial_i E)\big)$, $\delta\varphi$, $\frac{\partial\delta\varphi}{\partial\eta}$, $\frac{\partial^2\delta\varphi}{\partial\eta^2}$, there are three relations between invariants $\widetilde{\mathcal{I}}^k$, $k = 1, \dots, 6$,

$$\widetilde{\mathcal{I}}^2 = \frac{\partial\widetilde{\mathcal{I}}^1}{\partial\eta} = a\dot{\widetilde{\mathcal{I}}}^1 = U\widetilde{\mathcal{I}}^1, \quad \widetilde{\mathcal{I}}^4 = \frac{\partial\widetilde{\mathcal{I}}^3}{\partial\eta} = a\dot{\widetilde{\mathcal{I}}}^3 = V\widetilde{\mathcal{I}}^3, \quad \widetilde{\mathcal{I}}^6 = \frac{\partial\widetilde{\mathcal{I}}^5}{\partial\eta} = a\dot{\widetilde{\mathcal{I}}}^5 = W\widetilde{\mathcal{I}}^5, \quad U, V, W \text{ are } 9\times 9 \text{ matrices.} \tag{13}$$

Conclusion and References

Conclusion

General scalar perturbational invariant (1) was obtained in this research work. Based on this general invariant, we founded six linearly independent perturbational invariants (2-7). After that, we diagonalized the system of six linearly independent perturbational invariants and obtained the invariants (8). These diagonalized perturbational invariants were expressed as sums of perturbations of components of metric tensor and the corresponding functions of the scalar field $\delta\varphi$ which corresponds to a matter. At the end of this work, we presented the linearized relations (13) between diagonalized scalar perturbational invariants (8).

References

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