Coupled discrete solitonic equations and the periodic reduction

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Introduction

- Completely integrable 2D-lattices $\rightarrow$ periodic reduction $\rightarrow$ coupled completely integrable systems with branched dispersion [1], [2].
- Using the periodic reduction on a 2D-lattice for which the multi-solitons are known, one can easily construct the multi-solitons for the corresponding coupled systems.
- We are discussing a generalisation of the additive Bogoyavlensky equation (aB) to the multicomponent (matrix) case. The aB is an integrable semidiscrete generalized Volterra type equation [3]; a particular case - Lotka-Volterra [4], [5].
- The Hirota bilinear formalism $\rightarrow$ complete integrability [6].

The coupled semidiscrete aB system

The **coupled semidiscrete additive Bogoyavlensky system** with branched dispersion has the form:

$$\frac{d}{dt}Q_n(t) = Q_n\left(E_{\sigma_1}\sum_{j=1}^{N}Q_{n+j}(t)E_{\sigma_2} - E_{\sigma_2}\sum_{j=1}^{N}Q_{n-j}(t)E_{\sigma_1}\right), \qquad (1)$$

where $Q_n(t) = Q(n,t)$ is a diagonal matrix of complex functions $u_\nu(n,t)$, $\nu = \overline{1,M}$:

$$Q_n(t) = \begin{pmatrix} u_1(n,t) & 0 & 0 & \ldots\ldots & 0 \\ 0 & u_2(n,t) & 0 & \ldots\ldots & 0 \\ 0 & 0 & u_3(n,t) & \ldots\ldots & 0 \\ \ldots. & \ldots. & \ldots. & \ldots.. & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & u_M(n,t) \end{pmatrix}$$

and E_{σ_1} and E_{σ_2} are permutation matrices corresponding to the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ 2 & 3 & . & . & . & . & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ M & 1 & 2 & . & . & . & M-1 \end{pmatrix}.$$

The coupled semidiscrete Lotka-Volterra system

For any M and $N = 1$, on the components, system (1) becomes **coupled semidiscrete Lotka-Volterra system** [4], [5] and has the following expression:

$$\begin{aligned} \dot{u}_1 &= u_1(\overline{u_2} - \underline{u_M}) \\ \dot{u}_2 &= u_2(\overline{u_3} - \underline{u_1}) \\ &\ldots\ldots \\ \dot{u}_{M-1} &= u_{M-1}(\overline{u_M} - \underline{u_{M-2}}) \\ \dot{u}_M &= u_M(\overline{u_1} - \underline{u_{M-1}}) \end{aligned} \tag{2}$$

where:

$$u_\nu = u_\nu(n,t), \quad \overline{u_\nu} = u_\nu(n+1,t), \quad \underline{u_\nu} = u_\nu(n-1,t), \quad \nu = \overline{1, M}.$$

The Hirota bilinear form for coupled Lotka-Volterra

Starting from the coupled semidiscrete Lotka-Volterra sysytem (2) with any M coupled equations, and using the nonlinear substitution:

$$u_\nu(n,t) = 1 + \frac{\partial}{\partial t} \ln \frac{\overline{F_{\nu+1}}}{F_\nu}, \quad \nu = \overline{1, M} \tag{3}$$

where:

$$F_\nu = F_\nu(n,t), \quad \overline{F_\nu} = F_\nu(n+1,t),$$

$$\overset{(2)}{\overline{F_\nu}} = F_\nu(n+2,t), \quad \underset{(2)}{\underline{F_\nu}} = F_\nu(n-2,t).$$

The Hirota bilinear form for coupled Lotka-Volterra

we cast (2), which can be written in a compact manner as:

$$\dot{u}_\nu = u_1(\overline{u_{\nu+1}} - \underline{u_{\nu+M-1}}), \quad \nu = \overline{1, M} \tag{4}$$

into the Hirota bilinear form:

$$\mathbf{D_t} \overline{F_{\nu+1}} \cdot F_\nu + \overline{F_{\nu+1}} F_\nu = \overset{(2)}{\overline{F_{\nu+2}}}\, \underline{F_{\nu+M-1}}, \tag{5}$$

$F_\nu(n, t)$ - complex function, D_t - the Hirota bilinear operator [6]:

$$D_t^n a(t) \cdot b(t) = (\partial_t - \partial_{t'})^n a(t) b(t')|_{t=t'}. \tag{6}$$

The 1-soliton solutions for coupled Lotka-Volterra

Ansatz for 1-ss:

$$F_\nu = 1 + \epsilon_1^{\nu-1} e^{\eta_1}, \quad \nu = \overline{1, M}, \tag{7}$$

where:

$$\eta_1 = k_1 n + \omega_1 t + \eta_1^{(0)}$$

(k_1 - wave number, ω_1 - angular frequency, $\eta_1^{(0)}$ - arbitrary phase) with M possible branches of dispersion for the soliton:

$$\omega_1(k_1) = 2\left[\frac{\epsilon_1^2+1}{2\epsilon_1}\sinh k_1 + \frac{\epsilon_1^2-1}{2\epsilon_1}\cosh k_1\right], \ \epsilon_1 \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1, M}.$$

The 2-soliton solutions for coupled Lotka-Volterra

The 2-ss:

$$F_\nu = 1 + \epsilon_1^{\nu-1} e^{\eta_1} + \epsilon_2^{\nu-1} e^{\eta_2} + \epsilon_1^{\nu-1}\epsilon_2^{\nu-1} e^{\eta_1+\eta_2+A_{12}}, \quad \nu = \overline{1, M} \quad (8)$$

where:

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)}, \quad j = \overline{1,2}$$

the interaction phase:

$$e^{A12} = \frac{(e^{k_2}\epsilon_2 - e^{k_1}\epsilon_1)[e^{k_1}\epsilon_1(1+e^{k_1}\epsilon_1(1+e^{k_2}\epsilon_2)) - e^{k_2}\epsilon_2(1+e^{k_2}\epsilon_2(1+e^{k_1}\epsilon_1))]}{(e^{k_1}\epsilon_1 - 1)(e^{k_2}\epsilon_2 - 1)(e^{k_1+k_2}\epsilon_1\epsilon_2 - 1)^2}$$

with M possible branches of dispersion for each of the 2 solitons:

$$\omega_j(k_j) = 2\left[\frac{\epsilon_j^2+1}{2\epsilon_j}\sinh k_j + \frac{\epsilon_j^2-1}{2\epsilon_j}\cosh k_j\right], \epsilon_j \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1, M}, j = \overline{1,2}.$$

The multi-soliton solutions for coupled Lotka-Volterra

The $\mathcal{N}$-**soliton solution** for (2) has the expressions for F_ν, $(\nu = 1, ..., M)$:

$$F_\nu(n,t) = \sum_{\mu_1,..,\mu_\mathcal{N}=\{0,1\}} \exp\left(\sum_{i=1}^{\mathcal{N}} \mu_i[\eta_i + (\nu-1)\ln(\epsilon_i)] + \sum_{1\leq i<j}^{\mathcal{N}} \mu_i\mu_j A_{ij}\right), \quad (9)$$

where $\eta_j = k_j n + \omega_j t + \eta_j^{(0)}$, $\quad j = \overline{1,\mathcal{N}}$, and the interaction term has teh form:

$$e^{Aij} = \frac{(e^{k_j}\epsilon_j - e^{k_i}\epsilon_i)[e^{k_i}\epsilon_i(1+e^{k_i}\epsilon_i(1+e^{k_j}\epsilon_j)) - e^{k_j}\epsilon_j(1+e^{k_j}\epsilon_j(1+e^{k_i}\epsilon_i))]}{(e^{k_i}\epsilon_i - 1)(e^{k_j}\epsilon_j - 1)(e^{k_i+k_j}\epsilon_i\epsilon_j - 1)^2}$$

with the M branches of dispersion for each of the $\mathcal{N}$ solitons (k_j is the wave number of the j-soliton):

$$\omega_j(k_j) = 2\left[\frac{\epsilon_j^2+1}{2\epsilon_j}\sinh k_j + \frac{\epsilon_j^2-1}{2\epsilon_j}\cosh k_j\right], \ \epsilon_j \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1,M}, j = \overline{1,\mathcal{N}}.$$

The branches of dispersion are labelled by the index l. The parameter ϵ_j which characterizes the j-soliton ($j = \overline{1,\mathcal{N}}$) can have M values, the order M roots of unity.

The semidiscrete aB 2D-lattice

In order to solve the coupled semidiscrete aB system (1):

$$\frac{d}{dt}Q_n(t) = Q_n(t)\left(E_{\sigma_1}\sum_{j=1}^{N} Q_{n+j}(t)E_{\sigma_2} - E_{\sigma_2}\sum_{j=1}^{N} Q_{n-j}(t)E_{\sigma_1}\right),$$

one could start from the completely integrable **semidiscrete aB 2D-lattice** (in two discrete dimensions):

$$\frac{d}{dt}Q_{n,m}(t) = Q_{n,m}(t)\left(\sum_{j=1}^{N} Q_{n+j,m+j}(t) - \sum_{j=1}^{N} Q_{n-j,m-j}(t)\right) \quad (10)$$

Considering $Q(n,m,t)$ to be a periodic function only with respect to m and imposing periodic reduction on such coordinate in the 2D-lattice, one could obtain *coupled systems* of aB equations.

The periodic 2-reduction on aB 2D-lattice

Now lets consider the periodic 2-reduction on the m direction (meaning that $Q_{n,m}(t) = Q(n,m,t)$ is a periodic function only with respect to m and the period is 2). We omit writing the t dependency for simplicity. This means that:

$$Q(n,m) \equiv u_1(n),\ Q(n,m+1) \equiv u_2(n),$$

$$Q(n,m+2) \equiv u_1(n),\ Q(n,m-1) \equiv u_2(n),$$

Introducing this reduction in (10) and denoting:

$$u_1(n+1) = \overline{u_1}, \quad u_1(n+2) = \overset{(2)}{\overline{u_1}}, \quad u_1(n+N) = \overset{(N)}{\overline{u_1}}$$

$$u_1(n-1) = \underline{u_1}, \quad u_1(n-2) = \underset{(2)}{\underline{u_2}}, \quad u_1(n-N) = \underset{(N)}{\underline{u_1}}$$

The periodic 2-reduction on aB 2D-lattice

we get precisely (for even N):

$$\begin{aligned}\dot{u}_1 &= u_1\left(\overline{u_2}+\overline{u_1}^{(2)}+..+\overline{u_2}^{(N-1)}+\overline{u_1}^{(N)}-\underline{u_2}-\underset{(2)}{\underline{u_1}}-..-\underset{(N-1)}{\underline{u_2}}+\underset{(N)}{\underline{u_1}}\right)\\ \dot{u}_2 &= u_2\left(\overline{u_1}+\overline{u_2}^{(2)}+..+\overline{u_1}^{(N-1)}+\overline{u_2}^{(N)}-\underline{u_1}-\underset{(2)}{\underline{u_2}}-..-\underset{(N-1)}{\underline{u_1}}+\underset{(2)}{\underline{u_2}}\right).\end{aligned}$$

For $N=1$ we obtain coupled Lotka-Volterra system:

$$\begin{aligned}\dot{u}_1 &= u_1\left(\overline{u_2}-\underline{u_2}\right)\\ \dot{u}_2 &= u_2\left(\overline{u_1}-\underline{u_1}\right).\end{aligned}$$

The periodic 3-reduction on aB 2D-lattice

In the same way, if we impose periodic-3 reduction:

$$Q(n,m) \equiv u_1(n),\ Q(n,m+1) \equiv u_2(n),\ Q(n,m+2) \equiv u_3(n),$$
$$Q(n,m+3) \equiv u_1(n),\ Q(n,m-1) \equiv u_3(n),$$

we get the system with the following three coupled equations (for N multiple of three):

$$\begin{aligned}
\dot{u}_1 &= u_1 \left(\overline{u_2} + \overset{(2)}{\overline{u_3}} + \overset{(3)}{\overline{u_1}} + .. + \overset{(N)}{\overline{u_1}} - \underline{u_3} - \underset{(2)}{\underline{u_2}} - \underset{(3)}{\underline{u_1}} - .. - \underset{(N)}{\underline{u_1}} \right) \\
\dot{u}_2 &= u_2 \left(\overline{u_3} + \overset{(2)}{\overline{u_1}} + \overset{(3)}{\overline{u_2}} + .. + \overset{(N)}{\overline{u_2}} - \underline{u_1} - \underset{(2)}{\underline{u_3}} - \underset{(3)}{\underline{u_2}} - .. - \underset{(N)}{\underline{u_2}} \right) \\
\dot{u}_3 &= u_3 \left(\overline{u_1} + \overset{(2)}{\overline{u_2}} + \overset{(3)}{\overline{u_3}} + .. + \overset{(N)}{\overline{u_3}} - \underline{u_2} - \underset{(2)}{\underline{u_1}} - \underset{(3)}{\underline{u_3}} - .. - \underset{(N)}{\underline{u_3}} \right)
\end{aligned}$$

The periodic M-reduction on aB 2D-lattice

The coupled aB system comes out from the aB 2D-lattice equation (10) for any N, choosing a periodic M-reduction on m.

$$\begin{aligned}
\dot{u}_1 &= u_1 \left(\overline{u_2} + \overset{(2)}{\overline{u_3}} + .. + \overset{(N)}{\overline{u_{N+1}}} - \underline{u_M} - \underset{(2)}{\underline{u_{M-1}}} - .. - \underset{(N)}{\underline{u_{M-N+1}}} \right) \\
\dot{u}_2 &= u_2 \left(\overline{u_3} + \overset{(2)}{\overline{u_4}} + .. + \overset{(N)}{\overline{u_{N+2}}} - \underline{u_1} - \underset{(2)}{\underline{u_M}} - .. - \underset{(N)}{\underline{u_{M-N+2}}} \right) \\
\ldots &= \ldots\ldots\ldots\ldots \\
\dot{u}_{M-1} &= u_{M-1} \left(\overline{u_M} + \overset{(2)}{\overline{u_1}} + .. + \overset{(N)}{\overline{u_{M+N-1}}} - \underline{u_{M-2}} - \underset{(2)}{\underline{u_{M-3}}} - .. - \underset{(N)}{\underline{u_{M-N-1}}} \right) \\
\dot{u}_M &= u_M \left(\overline{u_1} + \overset{(2)}{\overline{u_2}} + .. + \overset{(N)}{\overline{u_{M+N}}} - \underline{u_{M-1}} - \underset{(2)}{\underline{u_{M-2}}} - .. - \underset{(N)}{\underline{u_{M-N}}} \right)
\end{aligned} \tag{11}$$

The Hirota bilinear form for aB 2D-lattice

Using the substitution (in this notation, m is not exponent):

$$Q_{n,m}(t) = 1 + \frac{\partial}{\partial t} log \frac{F_{n+1}^{m+1}}{F_n^m} \tag{13}$$

we cast the aB 2D-lattice, (10):

$$\frac{d}{dt} Q_{n,m}(t) = Q_{n,m}(t) \left(\sum_{j=1}^{N} Q_{n+j,m+j}(t) - \sum_{j=1}^{N} Q_{n-j,m-j}(t) \right)$$

into the Hirota bilinear form:

$$\mathbf{D_t} F_{n+1}^{m+1} \cdot F_n^m + F_{n+1}^{m+1} F_n^m = F_{n+1+N}^{m+1+N} F_{n-N}^{m-N}, \tag{14}$$

where F_n^m is a complex function, D_t is the Hirota bilinear operator.

The multi-soliton solutions for aB 2D-lattice

The 1-soliton solution is:

$$u_n^m = 1 + \frac{\partial}{\partial t} \log \frac{F_{n+1}^{m+1}}{F_n^m} = 1 + \frac{\partial}{\partial t} \log \frac{1 + e^{k_1(n+1)+p_1(m+1)+\omega_1 t+\eta_1^{(0)}}}{1 + e^{k_1 n+p_1 m+\omega_1 t+\eta_1^{(0)}}}$$

where:

$$F_n^m = 1 + e^{k_1 n+p_1 m+\omega_1 t+\eta_1^{(0)}}, \quad (\forall) \quad k_1, p_1 \in \mathbb{C}$$

and the dispersion relation has the form:

$$\omega_1 = 2\frac{\sinh \frac{(k_1+p_1)N}{2} \sinh \frac{(k_1+p_1)(N+1)}{2}}{\sinh \frac{k_1+p_1}{2}}.$$

The multi-soliton solution for aB 2D-lattice

For **2-soliton solution** we obtain:

$$F_n^m = 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}}, \quad \eta_j = k_j n+p_j m+\omega_j t+\eta_j^{(0)}, \quad j=1,2$$

the dispersion relation:

$$\omega_j = 2\frac{\sinh\frac{(k_j+p_j)N}{2}\sinh\frac{(k_j+p_j)(N+1)}{2}}{\sinh\frac{k_j+p_j}{2}},$$

and the interaction phase:

$$e^{A_{12}} = \frac{-\cosh\frac{k_1+p_1-k_2-p_2}{2}+\cosh\frac{(k_1+p_1-k_2-p_2)(1+2N)}{2}-(\omega_1-\omega_2)\sinh\frac{k_1+p_1-k_2-p_2}{2}}{-\cosh\frac{k_1+p_1+k_2+p_2}{2}-\cosh\frac{(k_1+p_1+k_2+p_2)(1+2N)}{2}+(\omega_1+\omega_2)\sinh\frac{k_1+p_1+k_2+p_2}{2}}.$$

The multi-soliton solution for aB 2D-lattice

For **3-soliton solution** we obtain:

$$F_n^m = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+A_{12}} + e^{\eta_1+\eta_3+A_{13}} + e^{\eta_2+\eta_3+A_{23}} + e^{\sum\limits_{i=1}^{3}\eta_i + \sum\limits_{1\le i<j}^{3} A_{ij}},$$

the dispersion relation:

$$\omega_j = 2\frac{\sinh\frac{(k_j+p_j)N}{2}\sinh\frac{(k_j+p_j)(N+1)}{2}}{\sinh\frac{k_j+p_j}{2}}, \quad j=\overline{1,3}$$

and the interaction phase:

$$e^{A_{ij}} = \frac{-\cosh\frac{k_i+p_i-k_j-p_j}{2} + \cosh\frac{(k_i+p_i-k_j-p_j)(1+2N)}{2} - (\omega_i-\omega_j)\sinh\frac{k_i+p_i-k_j-p_j}{2}}{-\cosh\frac{k_i+p_i+k_j+p_j}{2} - \cosh\frac{(k_i+p_i+k_j+p_j)(1+2N)}{2} + (\omega_i+\omega_j)\sinh\frac{k_i+p_i+k_j+p_j}{2}}.$$

The multi-soliton solutions for aB 2D-lattice

The $\mathcal{N}$-**soliton solutions** has the following form for F_n^m:

$$F_n^m(t) = \sum_{\mu_1,\mu_{\mathcal{N}}=\{0,1\}} \exp\left(\sum_{i=1}^{\mathcal{N}} \mu_i \eta_i + \sum_{1\leq i<j}^{\mathcal{N}} \mu_i \mu_j A_{ij}\right), \qquad (15)$$

where:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad j = \overline{1,\mathcal{N}},$$

$$\omega_j = 2\frac{\sinh \frac{(k_j+p_j)N}{2} \sinh \frac{(k_j+p_j)(N+1)}{2}}{\sinh \frac{k_j+p_j}{2}}, \quad j = \overline{1,\mathcal{N}}$$

$$e^{A_{ij}} = \frac{-\cosh \frac{k_i+p_i-k_j-p_j}{2} + \cosh \frac{(k_i+p_i-k_j-p_j)(1+2N)}{2} - (\omega_i - \omega_j)\sinh \frac{k_i+p_i-k_j-p_j}{2}}{-\cosh \frac{k_i+p_i+k_j+p_j}{2} - \cosh \frac{(k_i+p_i+k_j+p_j)(1+2N)}{2} + (\omega_i + \omega_j)\sinh \frac{k_i+p_i+k_j+p_j}{2}}.$$

The periodic reduction and the parallel between systems

Now, all the multi-soliton solutions for the coupled aB systems for any N are coming straightforward from the aB 2D-lattice (10) and one can easily see this by looking at the two bilinear forms:

- for aB (n,m,t) 2D-lattice:

$$\mathbf{D_t} F_{n+1}^{m+1} \cdot F_n^m + F_{n+1}^{m+1} F^m{}_n = F_{n+1+N}^{m+1+N} F_{n-N}^{m-N}, \tag{16}$$

- for coupled aB (n,t) system:

$$\mathbf{D_t} \overline{F_{\nu+1}} \cdot F_\nu + \overline{F_{\nu+1}} F_\nu = \overset{(N+1)}{\overline{F_{\nu+1+N}}} \underset{(N)}{\underline{F_{\nu-N}}}, \tag{17}$$

The systems are the same, considering that the second index, m, of $Q_{n,m}(t) = 1 + \frac{\partial}{\partial t} log \frac{F_{n+1}^{m+1}}{F_n^m}$ in (16) becomes $\nu = \overline{1, M}$ in (17), parameter which indiciates the soliton solutions $u_\nu(t) = 1 + \frac{\partial}{\partial t} log \frac{\overline{F_{\nu+1}}}{F_\nu}$ for the M-component aB system.

The periodic reduction and the parallel between systems

For example, in the case $M = 2$, the m-dependence is dropped, p_j appearing in the definitions will be $-\pi \mathrm{i}$, $+\pi \mathrm{i}$ making the dispersion relation to have two branches (allowing solitons to move either in the same direction or opposite to one another).

For $M = 3$, again the m-dependence is dropped, p_j will be $-2\pi \mathrm{i}/3$, $+2\pi \mathrm{i}/3$, $2\pi \mathrm{i}$ (its exponentials are the cubic roots of the unity), leading to the three branches of the dispersion relation.

For $\forall M$, dropping the m-dependence, $p_j \in \nu \frac{2\pi \mathrm{i}}{M}$, $\nu = 1, M$ (its exponentials are the M roots of unity), we have the M branches of dispersion.

Considering the above parallel, the periodic reduction proves again to be a very effective tool for deriving multi-soliton solution for multicomponent systems.

Conclusions

- In this paper we studied the coupled additive Bogoyavlensky system with branched dispersion relations and as a particular case ($N = 1$) the coupled Lotka-Volterra system;
- The main motivation was to see that the integrability survives in coupled systems.
- The main feature of such coupled systems is the structure of the dispersion relation (having multiple branches) and of the phases of the components, parametrised by the order M roots of unity. The existence of many branches of the dispersion relation allows more freedom in solitons interaction;
- It was shown by Hirota bilinear formalism that the coupled aB system is integrable and moreover it was shown that with a periodic reduction of an integrable aB 2D lattice equation the multi-solitons are easier to construct.

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