Fractional calculus in modelling hereditariness and nonlocality in transmission lines

Dušan Zorica

S. Cvetićanin, M. Rapaić

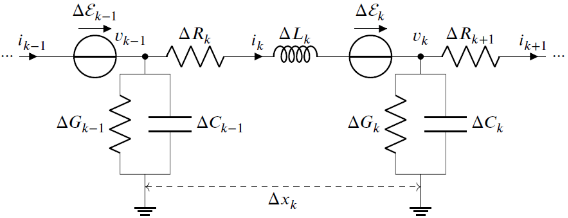
Department of Physics, Faculty of Sciences, University of Novi Sad
Mathematical Institute of the Serbian Academy of Arts and Sciences, Beograd

August 29, 2022.

BPU11 - Belgrade

Classical transmission line model - I

- Heaviside's elementary circuit



- Kirchhoff's laws yield

$$-v_{k-1}(t)+\Delta R_k(t)\, i_k(t)+\Delta L_k(t)\, i_k(t)-\Delta\mathcal{E}_k(t)+v_k(t)=0,$$

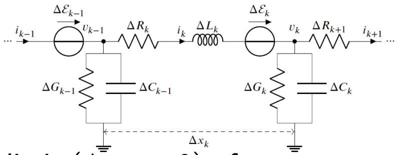
$$i_k(t)-i_{k+1}(t)-\Delta G_k(t)\, v_k(t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta C_k(t)\, v_k(t)\Big)=0,$$

- or, more precisely

$$-\frac{v_k(t)-v_{k-1}(t)}{\Delta x_k}=\frac{\Delta R_k(t)}{\Delta x_k}\, i_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\Delta L_k(t)}{\Delta x_k}\, i_k(t)-\frac{\Delta\mathcal{E}_k(t)}{\Delta x_k},$$

$$-\frac{i_{k+1}(t)-i_k(t)}{\Delta x}=\frac{\Delta G_k(t)}{\Delta x_k}\, v_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta C_k(t)}{\Delta x_k}\, v_k(t)\right).$$

Classical transmission line model - II



- The continuum limit ($\Delta x_k \to 0$) of

$$-\frac{v_k(t)-v_{k-1}(t)}{\Delta x_k}=\frac{\Delta R_k(t)}{\Delta x_k}\, i_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\Delta L_k(t)}{\Delta x_k}\, i_k(t)-\frac{\Delta\mathcal{E}_k(t)}{\Delta x_k},$$

$$-\frac{i_{k+1}(t)-i_k(t)}{\Delta x}=\frac{\Delta G_k(t)}{\Delta x_k}\, v_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta C_k(t)}{\Delta x_k}\, v_k(t)\right),$$

- yields classical telegrapher's equations

$$-\frac{\partial}{\partial x}v(x,t)=R\, i(x,t)+L\frac{\partial}{\partial t}i(x,t)-\mathcal{E}(x,t),$$

$$-\frac{\partial}{\partial x}i(x,t)=G\, v(x,t)+C\frac{\partial}{\partial t}v(x,t),$$

$$LC\frac{\partial^2}{\partial t^2}v(x,t)+(RC+LG)\frac{\partial}{\partial t}v(x,t)+RG\, v(x,t)-\frac{\partial^2}{\partial x^2}v(x,t)=-\frac{\partial}{\partial x}\mathcal{E}(x,t),$$

$$\frac{\partial^2}{\partial t^2}v(x,t)+\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)\frac{\partial}{\partial t}v(x,t)+\frac{1}{\tau_L\tau_C}v(x,t)-c^2\frac{\partial^2}{\partial x^2}v(x,t)=-c^2\frac{\partial}{\partial x}\mathcal{E}(x,t).$$

Classical telegrapher's equation

Classical telegrapher's equation ($x \in \mathbb{R},\ t > 0$)

$$\frac{\partial^2}{\partial t^2} v(x,t) + \left(\frac{1}{\tau_L} + \frac{1}{\tau_C}\right)\frac{\partial}{\partial t} v(x,t) + \frac{1}{\tau_L \tau_C} v(x,t) - c^2 \frac{\partial^2}{\partial x^2} v(x,t) = -c^2 \frac{\partial}{\partial x}\mathcal{E}(x,t),$$

represented by the system of equations

$$-\frac{\partial}{\partial x} v(x,t) = R\, i(x,t) + L \frac{\partial}{\partial t} i(x,t) - \mathcal{E}(x,t),$$
$$-\frac{\partial}{\partial x} i(x,t) = G\, v(x,t) + C \frac{\partial}{\partial t} v(x,t),$$

is subject to zero initial and boundary data

$$v(x,0) = 0, \quad i(x,0) = 0,$$
$$\lim_{x\to\pm\infty} v(x,t) = 0, \quad \lim_{x\to\pm\infty} i(x,t) = 0.$$

The solution to the classical telegrapher's equations is

$$v(x,t) = Q_R(x,t) *_{x,t} \mathcal{E}(x,t),$$

where Q_R is:

$$Q_R(x,t) = \frac{c^2}{\pi}\left(\int_0^{\xi_0} \frac{\sinh(\nu(\xi)\, t)}{\nu(\xi)}\, \xi \sin(\xi x)\, \mathrm{d}\xi + \int_{\xi_0}^{\infty} \frac{\sin(\omega(\xi)\, t)}{\omega(\xi)}\, \xi \sin(\xi x)\, \mathrm{d}\xi\right) \mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t},$$

with

$$\omega^2(\xi) = \left(c\xi + \frac{1}{2}\left|\frac{1}{\tau_L} - \frac{1}{\tau_C}\right|\right)\left(c\xi - \frac{1}{2}\left|\frac{1}{\tau_L} - \frac{1}{\tau_C}\right|\right) > 0, \ \textit{for}\ \xi \geq \xi_0 = \frac{1}{2c}\left|\frac{1}{\tau_L} - \frac{1}{\tau_C}\right|,$$
$$\nu^2(\xi) = -\,\omega^2(\xi), \ \textit{i.e.},\ \nu(\xi) = \omega(\xi), \ \textit{for}\ \xi < \xi_0.$$

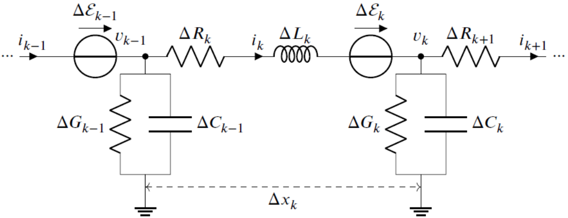
The integrand admits an asymtotics for fixed (x,t)

$$\frac{\sin(\omega(\xi)\, t)}{\omega(\xi)}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t}\, \xi \sin(\xi x) \sim \frac{1}{c}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t} \sin(\xi c t) \sin(\xi x) \to \infty \ \ \textit{as}\ \xi \to \infty,$$

$$\frac{\sinh(\nu(\xi)\, t)}{\nu(\xi)}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t}\, \xi \sin(\xi x) \sim x\, \frac{\sinh\left(\frac{1}{2}\left|\frac{1}{\tau_L} - \frac{1}{\tau_C}\right| t\right)}{\frac{1}{2}\left|\frac{1}{\tau_L} - \frac{1}{\tau_C}\right|}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t}\, \xi^2 \ \ \textit{as}\ \xi \to 0^+.$$

Non-local transmission line model - I

- Heaviside's elementary circuit



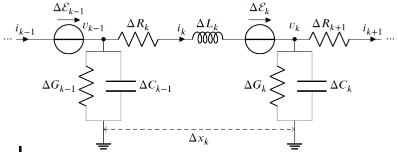
- Kirchhoff's laws yield

$$-v_{k-1}(t)+\Delta R_k(t)\, i_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\Delta\phi_k(t)-\Delta\mathcal{E}_k(t)+v_k(t)=0,$$

$$i_k(t)-i_{k+1}(t)-\Delta G_k(t)\, v_k(t)-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta C_k(t)\, v_k(t)\Big)=0,$$

$$\Delta\phi_k(t)=\Delta L_k(t)\, i_k(t)+\sum_{\substack{j=1\\ j\neq k}}^{N}\Delta^2 m_{kj}(t)\, i_j(t),$$

Non-local transmission line model - II



- or, more precisely

$$-\frac{v_k(t)-v_{k-1}(t)}{\Delta x_k}=\frac{\Delta R_k(t)}{\Delta x_k}\,i_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\Delta\phi_k(t)}{\Delta x_k}-\frac{\Delta\mathcal{E}_k(t)}{\Delta x_k},$$

$$-\frac{i_{k+1}(t)-i_k(t)}{\Delta x}=\frac{\Delta G_k(t)}{\Delta x_k}\,v_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta C_k(t)}{\Delta x_k}\,v_k(t)\right),$$

$$\frac{\Delta\phi_k(t)}{\Delta x_k}=\frac{\Delta L_k(t)}{\Delta x_k}\,i_k(t)+\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\Delta^2 m_{kj}(t)}{\Delta x_k\,\Delta x_j}\,i_j(t)\,\Delta x_j,$$

- implying, by the continuum limit ($\Delta x_k\to 0$), the non-local TEs

$$-\frac{\partial}{\partial x}v(x,t)=R(x,t)\,i(x,t)+\frac{\partial}{\partial t}\phi(x,t)-\mathcal{E}(x,t),$$

$$-\frac{\partial}{\partial x}i(x,t)=G(x,t)\,v(x,t)+\frac{\partial}{\partial t}\left(C(x,t)\,v(x,t)\right),$$

$$\phi(x,t)=L(x,t)\,i(x,t)+\int_a^b m(x,\zeta,t)\,i(\zeta,t)\,\mathrm{d}\zeta.$$

Non-local transmission line model - III

- The non-local TE expressed in terms of voltage is

$$
\begin{aligned}
LC\frac{\partial^2}{\partial t^2}v(x,t) &+ (RC+LG)\frac{\partial}{\partial t}v(x,t) + RG\,v(x,t) - \frac{\partial^2}{\partial x^2}v(x,t) \\
&- G\int_a^b\int_a^\eta \frac{\partial}{\partial x}m(x,\eta)\frac{\partial}{\partial t}v(\zeta,t)\,\mathrm{d}\zeta\mathrm{d}\eta \\
&- C\int_a^b\int_a^\eta \frac{\partial}{\partial x}m(x,\eta)\frac{\partial^2}{\partial t^2}v(\zeta,t)\,\mathrm{d}\zeta\mathrm{d}\eta \\
&\qquad = -\frac{\partial}{\partial x}\mathcal{E}(x,t) + \frac{\partial}{\partial t}i(a,t)\int_a^b \frac{\partial}{\partial x}m(x,\eta)\,\mathrm{d}\eta.
\end{aligned}
$$

Non-local telegrapher's equation

Non-local telegrapher's equations ($x \in \mathbb{R}$, $t > 0$)

$$-\frac{\partial}{\partial x} v(x,t) = R\, i(x,t) + \frac{\partial}{\partial t}\phi(x,t) - \mathcal{E}(x,t),$$
$$-\frac{\partial}{\partial x} i(x,t) = G\, v(x,t) + C\frac{\partial}{\partial t} v(x,t),$$
$$\phi(x,t) = L\, i(x,t) + m(|x|) *_x i(x,t),$$

with

$$m(|x|) *_x i(x,t) = M\int_{-\infty}^{\infty} \bar{m}(|x-\zeta|)\, i(\zeta,t)\,\mathrm{d}\zeta,$$

are subject to zero initial and boundary data

$$v(x,0) = 0, \quad i(x,0) = 0,$$
$$\lim_{x\to\pm\infty} v(x,t) = 0, \quad \lim_{x\to\pm\infty} i(x,t) = 0.$$

Non-local telegrapher's equation

The cross-inductivity kernels are chosen among

Table: Several choices of cross-inductivity kernels.

Kernel type	$m(\lvert x-\zeta\rvert)$	$M\delta(\lvert x-\zeta\rvert)$
Power	$\frac{M}{2\Gamma(\alpha)}\frac{\lvert x-\zeta\rvert^{\alpha-1}}{\ell^{\alpha}}$	*as* $\alpha\to 0$
Exponential	$\frac{M}{2\ell}\mathrm{e}^{-\frac{\lvert x-\zeta\rvert}{\ell}}$	*as* $\ell\to 0$
Gaussian	$\frac{M}{\ell\sqrt{\pi}}\mathrm{e}^{-\frac{\lvert x-\zeta\rvert^2}{\ell^2}}$	*as* $\ell\to 0$

Model parameters are: cross-inductivity per-unit-length M*, non-locality parameter (characteristic length)* ℓ*, and* $\alpha\in(0,1)$*.*

The solution to the non-local telegrapher's equations is

$$v(x,t) = Q(x,t) *_{x,t} \mathcal{E}(x,t),$$

where Q is:

$$\begin{aligned} Q(x,t) =& \frac{c^2}{\pi} \left(\int_0^{\xi_1} \frac{\sinh(\nu(\xi)\,t)}{\nu(\xi)} \,\mathrm{e}^{-\mu(\xi)\,t} \, \frac{\xi \sin(\xi x)}{1 + \frac{\tau_M}{\tau_L} \tilde{\tilde{m}}(|\xi|)} \,\mathrm{d}\xi \right. \\ & \left. + \int_{\xi_1}^{\xi_2} f(\xi,t) \,\mathrm{e}^{-\mu(\xi)\,t} \, \frac{\xi \sin(\xi x)}{1 + \frac{\tau_M}{\tau_L} \tilde{\tilde{m}}(|\xi|)} \,\mathrm{d}\xi + \int_{\xi_2}^{\infty} \frac{\sin(\omega(\xi)\,t)}{\omega(\xi)} \,\mathrm{e}^{-\mu(\xi)\,t} \, \frac{\xi \sin(\xi x)}{1 + \frac{\tau_M}{\tau_L} \tilde{\tilde{m}}(|\xi|)} \,\mathrm{d}\xi \right), \end{aligned}$$

with

$$\begin{aligned} \mu(\xi) &= \frac{1}{2} \left(\frac{1}{\tau_L + \tau_M \tilde{\tilde{m}}(|\xi|)} + \frac{1}{\tau_C} \right), \\ \nu^2(\xi) &= \left(\frac{1}{2} \left(\frac{1}{\tau_L + \tau_M \tilde{\tilde{m}}(|\xi|)} - \frac{1}{\tau_C} \right) \right)^2 - \frac{(c\xi)^2}{1 + \frac{\tau_M}{\tau_L} \tilde{\tilde{m}}(|\xi|)}, \; c = \frac{1}{K\sqrt{\tau_L \tau_C}}, \\ f(\xi,t) &= \frac{\mathrm{e}^{\nu(\xi)\,t} - \mathrm{e}^{-\nu(\xi)\,t}}{2\nu(\xi)} = \begin{cases} \dfrac{\sinh(\nu(\xi)\,t)}{\nu(\xi)}, & \textit{if } \nu^2(\xi) \geq 0, \\[2ex] \dfrac{\sin(\omega(\xi)\,t)}{\omega(\xi)}, & \textit{if } \nu^2(\xi) < 0, \textit{ with } \omega(\xi) = -\mathrm{i}\nu(\xi). \end{cases} \end{aligned}$$

The integrand admits an asymtotics for fixed (x, t)

in the case of all considered kernels

$$\frac{\sin(\omega(\xi)\,t)}{\omega(\xi)}\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\tilde{m}}(|\xi|)}\sim\frac{1}{c}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_C}\right)t}\sin(\xi ct)\sin(\xi x)\quad \text{as } \xi\to\infty,$$

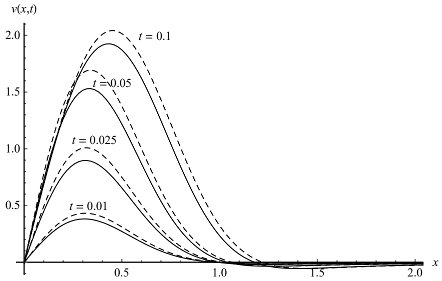
in the case of the power-type kernel

$$\frac{\sinh(\nu(\xi)\,t)}{\nu(\xi)}\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\tilde{m}}(|\xi|)}\sim\frac{\tau_L}{\tau_M}\,\frac{\ell^\alpha x}{\cos\frac{\alpha\pi}{2}}\,\frac{\sinh\frac{t}{2\tau_C}}{\frac{1}{2\tau_C}}\,\mathrm{e}^{-\frac{t}{2\tau_C}}\,\xi^{2+\alpha}\quad \text{as } \xi\to 0^+$$

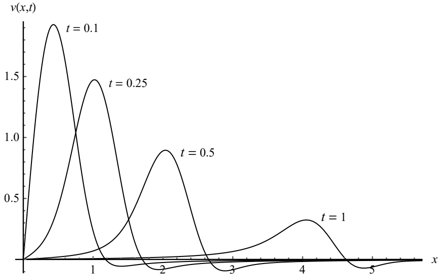
in the case of the Gauss- and exponential-type kernel

$$\frac{\sinh(\nu(\xi)\,t)}{\nu(\xi)}\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\tilde{m}}(|\xi|)}\sim\frac{\tau_L}{\tau_M}\,\frac{\ell^\alpha x}{\cos\frac{\alpha\pi}{2}}\,\frac{\sinh\frac{t}{2\tau_C}}{\frac{1}{2\tau_C}}\,\mathrm{e}^{-\frac{t}{2\tau_C}}\,\xi^{2+\alpha}\quad \text{as } \xi\to 0^+.$$

Spatial profiles - power-type kernel - I



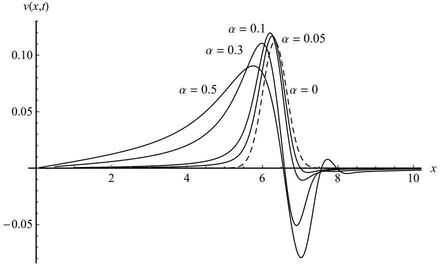
(a) Spatial profiles at time-instances near the initial one.



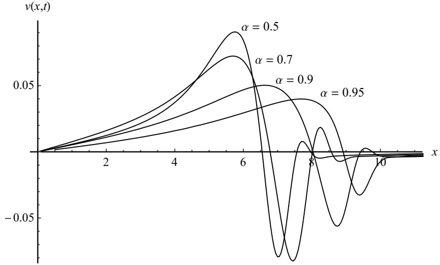
(b) Spatial profiles at later time-instances.

Figure: Power type cross-inductivity kernel – time evolution of spatial profile of mollified impulse response, obtained analytically (solid line) and numerically (dashed line), for model parameters: $K = 0.5$, $\tau_C = 0.5$, $\tau_L = 0.2$, $\tau_M = 0.25$, $\ell = 0.1$, $\alpha = 0.25$, with $\varepsilon = 0.05$ and $\epsilon = 0.002$.

Spatial profiles - power-type kernel - II



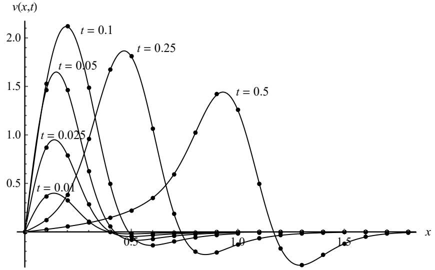
(a) Spatial profiles for smaller values of fractionalization parameter.



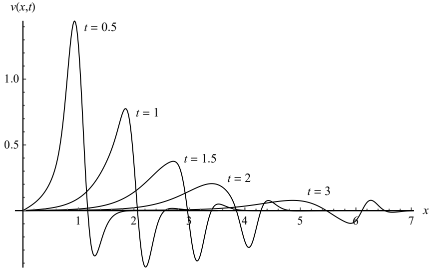
(b) Spatial profiles for larger values of fractionalization parameter.

Figure: Power type cross-inductivity kernel – spatial profiles of mollified impulse response at $t = 1.5$, depending on fractional differentiation order α, for model parameters: $K = 0.5$, $\tau_C = 0.5$, $\tau_L = 0.2$, $\tau_M = 0.25$, $\ell = 0.1$, with $\varepsilon = 0.05$, depicted by solid line for $\alpha \in (0, 1)$ and dashed line for $\alpha = 0$.

Spatial profiles - exponential-type kernel - I



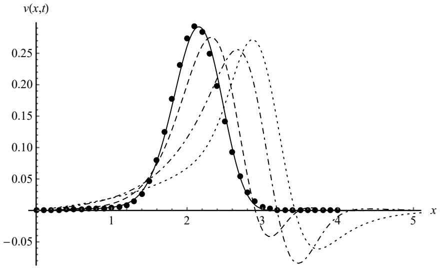
(a) Spatial profiles at time-instances near the initial one.



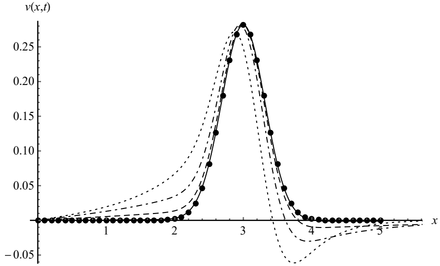
(b) Spatial profiles at later time-instances.

Figure: Exponential type cross-inductivity kernel – time evolution of spatial profile of mollified impulse response, obtained analytically (solid line) and numerically (dots), for model parameters: $K = 0.5$, $\tau_C = 1$, $\tau_L = 1$, $\tau_M = 1$, $\ell = 0.25$, with $\varepsilon = 0.01$ and $\epsilon = 0.002$.

Spatial profiles - exponential-type kernel - II



(a) Solid, dashed, dot-dashed, and dotted lines correspond to non-locality parameters $\ell \in \{0.05, 0.15, 0.4, 0.8\}$; dots correspond to $\ell = 0$.

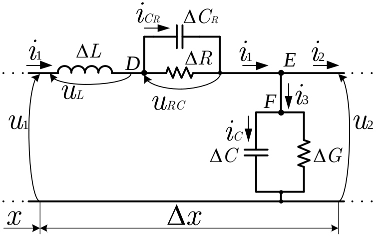


(b) Dotted, dot-dashed, dashed, and solid lines correspond to non-locality parameters $\ell \in \{0.8, 1.5, 3, 10\}$; dots correspond to $\ell \to \infty$.

Figure: Exponential type cross-inductivity kernel – spatial profiles of mollified impulse response at $t = 1.5$, depending on non-locality parameter ℓ, for model parameters: $K = 0.5$, $\tau_C = 1$, $\tau_L = 1$, $\tau_M = 1$, with $\varepsilon = 0.05$.

Hereditary transmission line model - I

- Heaviside's elementary circuit



- Constitutive models

$$\phi(t)=\Delta L_{\xi}\,{}_0\mathrm{I}_t^{1-\xi} i_L(t) \text{ and } q(t)=\Delta C_{\xi}\,{}_0\mathrm{I}_t^{1-\xi} u_C(t),\ \xi\in(0,1),$$

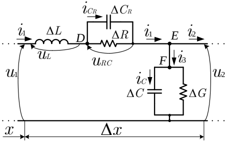
$${}_0\mathrm{I}_t^{\xi} f(t)=\frac{1}{\Gamma(\xi)}\int_0^t (t-\tau)^{\xi-1} f(\tau)\mathrm{d}\tau=\frac{t^{\xi-1}}{\Gamma(\xi)}*f(t),\ \xi>0, \text{ imply}$$

$$u_L(t)=\frac{\mathrm{d}}{\mathrm{d}t}\phi(t)=\Delta L_{\xi}\,\frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\xi} i_L(t)=\Delta L_{\xi}\,{}_0\mathrm{D}_t^{\xi} i_L(t), \text{ and}$$

$$i_C(t)=\frac{\mathrm{d}}{\mathrm{d}t}q(t)=\Delta C_{\xi}\,\frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\xi} u_C(t)=\Delta C_{\xi}\,{}_0\mathrm{D}_t^{\xi} u_C(t), \text{ where}$$

$${}_0\mathrm{D}_t^{\xi} f(t)=\frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\xi} f(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{-\xi}}{\Gamma(1-\xi)}*f(t)\right).$$

Hereditary transmission line model - II



- Kirchhoff's laws yield

$$u(x+\Delta x,t)-u(x,t)=-\Delta L\, {}_0\mathrm{D}_t^{\alpha} i(x,t)-u_{RC}(x,t),$$
$$i(x+\Delta x,t)-i(x,t)=-\Delta C\, {}_0\mathrm{D}_t^{\gamma} u(x+\Delta x,t)-\Delta G u(x+\Delta x,t),$$
$$\Delta R i(x,t)=u_{RC}(x,t)+\Delta R \Delta C_R\, {}_0\mathrm{D}_t^{\beta} u_{RC}(x,t),$$

- or, more precisely

$$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}=-\frac{\Delta L}{\Delta x}\, {}_0\mathrm{D}_t^{\alpha} i(x,t)-\frac{u_{RC}(x,t)}{\Delta x},$$
$$\frac{i(x+\Delta x,t)-i(x,t)}{\Delta x}=-\frac{\Delta C}{\Delta x}\, {}_0\mathrm{D}_t^{\gamma} u(x+\Delta x,t)-\frac{\Delta G}{\Delta x} u(x+\Delta x,t),$$
$$\frac{\Delta R}{\Delta x} i(x,t)=\frac{u_{RC}(x,t)}{\Delta x}+\Delta R \Delta C_R\, {}_0\mathrm{D}_t^{\beta} \frac{u_{RC}(x,t)}{\Delta x}.$$

Hereditary transmission line model - III

- The continuum limit ($\Delta x_k \to 0$) of

$$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}=-\frac{\Delta L}{\Delta x}\,{}_0\mathrm{D}_t^{\alpha} i(x,t)-\frac{u_{RC}(x,t)}{\Delta x},$$
$$\frac{i(x+\Delta x,t)-i(x,t)}{\Delta x}=-\frac{\Delta C}{\Delta x}\,{}_0\mathrm{D}_t^{\gamma} u(x+\Delta x,t)-\frac{\Delta G}{\Delta x} u(x+\Delta x,t),$$
$$\frac{\Delta R}{\Delta x} i(x,t)=\frac{u_{RC}(x,t)}{\Delta x}+\Delta R\Delta C_R\,{}_0\mathrm{D}_t^{\beta}\frac{u_{RC}(x,t)}{\Delta x},$$

- yields hereditary telegrapher's equations

$$\frac{\partial}{\partial x} u(x,t)=-L\,{}_0\mathrm{D}_t^{\alpha} i(x,t)-u'(x,t),$$
$$\frac{\partial}{\partial x} i(x,t)=-C\,{}_0\mathrm{D}_t^{\gamma} u(x,t)-Gu(x,t),$$
$$Ri(x,t)=u'(x,t)+\tau\,{}_0\mathrm{D}_t^{\beta} u'(x,t),$$

i.e., hereditary TE expressed in terms of voltage

$$\Big(\tau LC\,{}_0\mathrm{D}_t^{\alpha+\beta+\gamma}+\tau LG\,{}_0\mathrm{D}_t^{\alpha+\beta}+LC\,{}_0\mathrm{D}_t^{\alpha+\gamma}+LG\,{}_0\mathrm{D}_t^{\alpha}+RC\,{}_0\mathrm{D}_t^{\gamma}+RG\Big)u(x,t)=\Big(\tau\,{}_0\mathrm{D}_t^{\beta}+1\Big)\frac{\partial^2}{\partial x^2}u(x,t).$$

Hereditary telegrapher's equation

Hereditary telegrapher's equation ($x \in [0, \infty)$ $t > 0$)

$$\left(\tau LC\,{}_0\mathrm{D}_t^{\alpha+\beta+\gamma} + \tau LG\,{}_0\mathrm{D}_t^{\alpha+\beta} + LC\,{}_0\mathrm{D}_t^{\alpha+\gamma} + LG\,{}_0\mathrm{D}_t^{\alpha} + RC\,{}_0\mathrm{D}_t^{\gamma} + RG\right) u\left(x,t\right) = \left(\tau\,{}_0\mathrm{D}_t^{\beta} + 1\right) \frac{\partial^2}{\partial x^2} u\left(x,t\right).$$

represented by the system of equations

$$\frac{\partial}{\partial x} u\left(x,t\right) = -L\,{}_0\mathrm{D}_t^{\alpha} i\left(x,t\right) - u'\left(x,t\right),$$

$$\frac{\partial}{\partial x} i\left(x,t\right) = -C\,{}_0\mathrm{D}_t^{\gamma} u\left(x,t\right) - Gu\left(x,t\right),$$

$$Ri\left(x,t\right) = u'\left(x,t\right) + \tau\,{}_0\mathrm{D}_t^{\beta} u'\left(x,t\right),$$

is subject to zero initial and boundary data

$$v(x,0) = 0, \quad i(x,0) = 0, \quad i'(x,0) = 0,$$

$$u(0,t) = u_0(t), \quad \lim_{x\to\infty} u(x,t) = 0.$$

The solution to the hereditary telegrapher's equations is

$$u(x,t) = u_\delta(x,t) *_t \; u_0(t),$$

where u_δ is:

$$u_\delta^{(\mathrm{I,II})}(x,t) = \frac{1}{\pi}\int_0^\infty \sin\left(\mathrm{Im}\, k\left(\rho \mathrm{e}^{\mathrm{i}\pi}\right) x\right) \mathrm{e}^{-\left(\rho t + \mathrm{Re}\, k\left(\rho \mathrm{e}^{\mathrm{i}\pi}\right) x\right)} \mathrm{d}\rho,$$
$$u_\delta^{(\mathrm{III})}(x,t) = \frac{1}{\pi}\int_0^\infty \sin\left(\rho t \sin\varphi_0 - \mathrm{Im}\, k\left(\rho \mathrm{e}^{\mathrm{i}\varphi_0}\right) x + \varphi_0\right) \mathrm{e}^{-\left(\rho t |\cos\varphi_0| + \mathrm{Re}\, k\left(\rho \mathrm{e}^{\mathrm{i}\varphi_0}\right) x\right)} \mathrm{d}\rho,$$

with the propagation coefficient

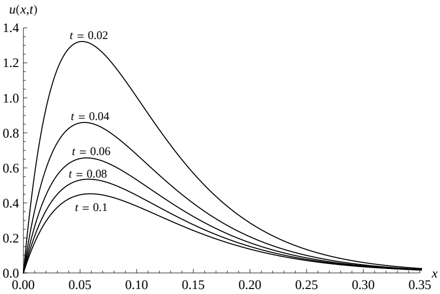
$$k(s) = \sqrt{\frac{(RL_\alpha C_\beta s^{\alpha+\beta} + L_\alpha s^\alpha + R)(C_\gamma s^\gamma + G)}{RC_\beta s^\beta + 1}}.$$

Case I - propagation coefficient k, except for $s = 0$, does not have any other branching points,

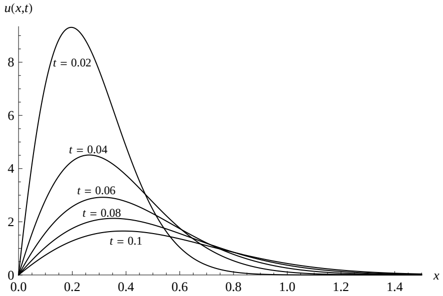
Case II - propagation coefficient k, except for $s = 0$, has a negative real branching point,

Case III - propagation coefficient k, except for $s = 0$, has a pair of complex conjugated branching points $s_0 = \rho_0 \mathrm{e}^{\mathrm{i}\varphi_0}$ and $\bar{s}_0$.

Diffusion-like spatial profiles - I



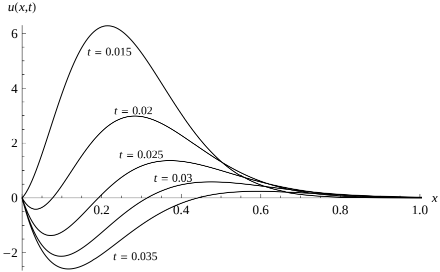
(a) $\alpha = \frac{1}{4}$, $\beta = \gamma = \frac{2}{3}$, $a = 2 \cdot 9^{1/3}$ and $b = 450$ - NBP



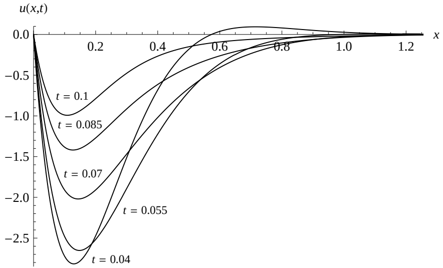
(b) $\alpha = \frac{2}{3}$, $\beta = \frac{5}{6}$, $\gamma = \frac{1}{4}$, $a = 2 \cdot 3^{1/3}$ and $b = 3$ - NRBP

Figure: Impulse response $u(x,t)$ as a function of position x at discrete time instants t.

Diffusion-like spatial profiles - II



(a)



(b)

Figure: Impulse response $u(x,t)$ as a function of position x at discrete time instants t for $\alpha = \frac{2}{3}$, $\beta = \frac{5}{6}$, $\gamma = \frac{1}{4}$, $a = 2 \cdot 3^{1/3}$ and $b = 300$ - CCBP.

Wave-like spatial profiles - I

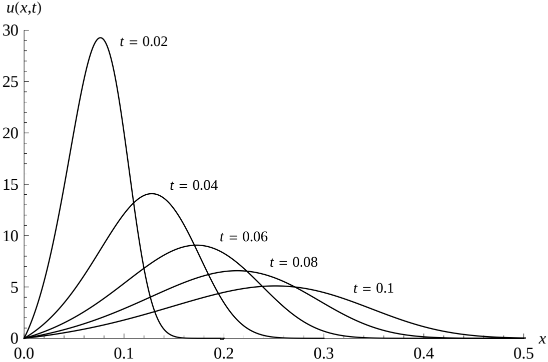


Figure: Impulse response $u(x,t)$ as a function of position x at discrete time instants t for $\alpha = \frac{5}{6}$, $\beta = \gamma = \frac{2}{3}$, $a = 2 \cdot 9^{1/3}$ and $b = 4.5$ - NBP.

Wave-like spatial profiles - II

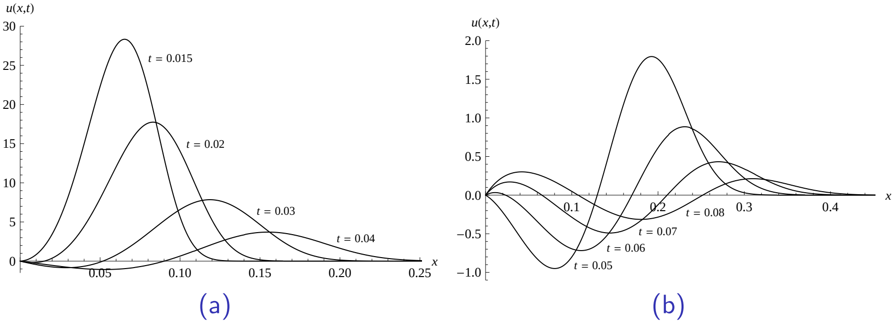


Figure: Impulse response $u(x,t)$ as a function of position x at discrete time instants t for $\alpha = \frac{5}{6}$, $\beta = \gamma = \frac{2}{3}$, $a = 2 \cdot 9^{1/3}$ and $b = 450$ - CCBP.