Thermodynamic Equilibrium, Nambu Brackets and Induced Hessians

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The 11th Conference of the Balkan Physical Union (BPU11 Congress)

28 August – 1 September 2022
Belgrade, Serbia

Outline

1. The physics behind the first law of thermodynamics
2. Conditions for global and local thermodynamic equilibrium
3. Change of variables, Nambu brackets and induced Hessians
4. Applications to black hole physics and holography

The First Law of Thermodynamics

Perturbations of the thermal equilibrium states satisfy the first law of thermodynamics:

$$dU = TdS - pdV + \mu dN + \cdots = \sum_{a=1}^{n} I_a dE^a. \tag{1}$$

- $I_a = (T, -p, \mu, ...)$ – intensive state quantities.
- $E^a = (S, V, N, ...)$ – extensive state variables.
- $U = U(E^a)$ – the energy potential in its natural variables.

In equilibrium $U = U(E^a)$ is the fundamental relation and EoS:

$$I_a = \left.\frac{\partial U}{\partial E^a}\right|_{E^1,\ldots,\hat{E}^a,\ldots,E^n} \quad \Rightarrow \quad T = \left.\frac{\partial U}{\partial S}\right|_{V,N}, \quad -p = \left.\frac{\partial U}{\partial V}\right|_{S,N}, \ldots \tag{2}$$

Quasi-homogeneous functions

U is quasi-homogeneous of degree r and type $(r_1, ..., r_n)$ under dilatations by a scale factor $\lambda > 0$ if

$$U(\lambda^{r_1} S, \lambda^{r_2} V, \lambda^{r_3} N_1, ..., \lambda^{r_n} N_k) = \lambda^r U(S, V, N_1, ..., N_k). \qquad (3)$$

A differentiable quasi-homogeneous function U satisfies a generalized Euler identity (necessary and sufficient):

$$\left(r_1 E^1 \frac{\partial}{\partial E^1} + r_2 E^2 \frac{\partial}{\partial E^2} + \cdots + r_n E^n \frac{\partial}{\partial E^n} \right) U = rU. \qquad (4)$$

In equilibrium Euler's identity becomes:

$$\sum_{a=1}^{n} r_a I_a E^a = rU. \qquad (5)$$

This is the Smarr relation for black holes.

Global Thermodynamic Stability in U Representation

Global minimum of the energy potential (fully convex) iff

$$d^2U = dE^T.(Hess_E U).dE \geq 0 \ \forall E, \quad [Hess_E U(E)]_{ab} = \frac{\partial^2 U(E)}{\partial E^a \partial E^b}. \tag{6}$$

- An eigenvalue problem: $\lambda_i \geq 0$,

$$Hess_E U(E) = M. \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} .M^{-1}. \tag{7}$$

- Alternatively: Sylvester's criterion for positive semi-definite quadratic forms: $\Delta_k \geq 0$, $k = 1, ..., n$. [Phys. Rev. D **105**, no. 4, 044033 (2022)].

Global Thermodynamic Stability in S Representation

Global maximum of the entropy potential (fully concave) iff

$$d^2S = dE^T.(Hess_E S).dE \leq 0 \; \forall E, \quad [Hess_E S(E)]_{ab} = \frac{\partial^2 S(E)}{\partial E^a \partial E^b}. \quad (8)$$

- An eigenvalue problem: $s_i \leq 0$,

$$Hess_E S(E) = N. \begin{pmatrix} s_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix} .N^{-1}. \quad (9)$$

- Alternatively: Sylvester's criterion for negative semi-definite quadratic forms: $(-1)^k \Delta_k \geq 0$. $k = 1, ..., n$.

Legendre Transformation

Let $\mathcal{P}(I) = \{\mathcal{P}_1(I), \mathcal{P}_2(I), ...\} = \{\{I_a\}, \{I_a, I_b\}, ...\}$ is the power set of all intensive variables. Define a dot product of power sets:

$$\mathcal{P}(I).\mathcal{P}(E) := \{\{I_a\}, \{I_a, I_b\}, ...\}. \begin{Bmatrix} \{E^a\} \\ \{E^a, E^b\} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} I_a E^a \\ I_a E^a + I_b E^b \\ \vdots \end{Bmatrix}, \quad (10)$$

$$\mathcal{P}_k(I).\mathcal{P}_k(E) = I_a E^a + I_b E^b + I_c E^c + \cdots \quad (11)$$

- Legendre transformation to a new potential Φ:

$$\Phi = \mathcal{L}_{\mathcal{P}_k(E)} U(E) := U(E) - \mathcal{P}_k(I).\mathcal{P}_k(E). \quad (12)$$

- Masseue-Plank potentials Ψ:

$$\Psi = \mathcal{L}_{\mathcal{P}_k(E)} S(E) := S(E) - \mathcal{P}_k(I).\mathcal{P}_k(E). \quad (13)$$

Energy/Entropy Derived Potentials

Energy derived:

$$\mathcal{L}_S U = U - TS = F\,(Helmholtz), \tag{14}$$

$$\mathcal{L}_V U = U + PV = H\,(Enthalpy), \tag{15}$$

$$\mathcal{L}_{S,V} U = U - TS + PV = G\,(Gibbs), \tag{16}$$

$$\mathcal{L}_{S,V,N} U = U - TS + PV - \mu N = \Omega\,(Grand). \tag{17}$$

Entropy derived (Massieu-Planck):

$$\mathcal{L}_U S = S - \frac{U}{T} = \Phi\,(Massieu), \tag{18}$$

$$\mathcal{L}_V S = S - \frac{PV}{T}, \tag{19}$$

$$\mathcal{L}_{U,V} S = S - \frac{U}{T} - \frac{PV}{T} = \Xi\,(Planck), \tag{20}$$

$$\mathcal{L}_{U,V,N} S = S - \frac{U}{T} - \frac{PV}{T} + \frac{\mu N}{T}. \tag{21}$$

Local Thermodynamic Equilibrium

Let X^m is a set of m fix state variables and Y^{m+1} is a set of $m+1$ arbitrary state quantities (also containing elements of X^m):

- Specific heats (Davies'77, Mansoori & Mirza'14):

$$C_X = T\left(\frac{\partial S}{\partial T}\right)\Bigg|_X = T\frac{\{S,X\}_Y}{\{T,X\}_Y}. \tag{22}$$

- Local stability conditions and phase transitions (Davies'77):

$$C_X = \begin{cases} >0, & \text{locally stable,} \\ <0, & \text{locally unstable,} \\ =0, & \text{phase transition,} \\ \to \pm\infty & \text{phase transition.} \end{cases} \tag{23}$$

Applications to Black Holes and Holography

VA, HD, MR, RR and TV (upcoming 2022)

System	Globally stable	Locally stable
Schwarzschild	No	No
Reisner-Nordstrom	No	Yes
Kerr	No	Yes
Kerr-Neuman	No	Yes
Kerr-Neuman-AdS	Yes	Yes

Phys. Rev. D **105**, no.4, 044033 (2021).

Phys. Rev. D **99**, no.12, 126007 (2019).

Induced Hessians and Information Geometry

- Generalized pushforward from e-space to E-space (Avramov, Dimov, Iliev, Radomirov, Rashkov, Vetsov upcoming 2022):

$$g_{ab}^{(W)}(E) = \frac{\partial^2 U(E)}{\partial E^a \partial E^b} = \frac{\partial e^\alpha}{\partial E^a}\left(\frac{\partial^2 U(e)}{\partial e^\alpha \partial e^\beta} - \Gamma^\gamma_{\alpha\beta}(e)\frac{\partial U(e)}{\partial e^\gamma}\right)\frac{\partial e^\beta}{\partial E^b}. \quad (24)$$

- Generalized pullback from E-space to e-space:

$$g_{\alpha\beta}^{(W)}(e) = \frac{\partial^2 U(e)}{\partial e^\alpha \partial e^\beta} = \left(\frac{\partial e^\alpha}{\partial E^a}\right)^{-1}\left(\frac{\partial^2 U(E)}{\partial E^a \partial E^b}\right)\left(\frac{\partial e^\beta}{\partial E^b}\right)^{-1} + \Gamma^\gamma_{\alpha\beta}(e)\frac{\partial U(e)}{\partial e^\gamma}. \quad (25)$$

- Induced metric:

$$h_{\alpha\beta}(e) = \delta_{ab}\frac{\partial E^a}{\partial e^\alpha}\frac{\partial E^b}{\partial e^\beta}, \quad \Gamma^\gamma_{\alpha\beta}(e) = \frac{1}{2}h^{\gamma\delta}(\partial_\beta h_{\delta\alpha} + \partial_\alpha h_{\delta\beta} - \partial_\delta h_{\alpha\beta}). \quad (26)$$

Conjectures and Other Relations

- Fluctuation theory (G. Ruppeiner '75)
- Linear stability of black holes = thermodynamic stability (R. Wald '12)
- Geometrothermodynamics (H. Quevedo '07)
- Probing quantum gravity (G. Rupeiner '10)
- Holographic complexity (D. Stanford, L. Susskind '14)

Probing Quantum Gravity

G. Rupeiner '10:

$$R = \begin{cases} > 0, & \text{Elliptic geometry} \to \text{repulsive interactions}, \\ < 0, & \text{Hyperbolic geometry} \to \text{attractive interactions}, \\ = 0, & \text{Flat geometry} \to \text{free system}, \\ \to \pm\infty & \text{Phase transition}. \end{cases} \tag{27}$$

What about higher order algebraic invariants?

And even more about the pletora of differential invariants?

Relations to Nambu Brackets

VA, HD, MR, RR and TV (2022):

$$\frac{\partial^2 E}{\partial S^2}\Big|_{J,V,\alpha} = \frac{\partial T}{\partial S}\Big|_{J,V,\alpha} = \frac{\{T, J, V, \alpha\}_{T,\Omega,P,\alpha}}{\{S, J, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (28)$$

$$\frac{\partial^2 E}{\partial J^2}\Big|_{S,V,\alpha} = \frac{\partial \Omega}{\partial J}\Big|_{S,V,\alpha} = \frac{\{\Omega, S, V, \alpha\}_{T,\Omega,P,\alpha}}{\{J, S, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (29)$$

$$\frac{\partial^2 E}{\partial J \partial S}\Big|_{V,\alpha} = \frac{\partial}{\partial J}\left(\frac{\partial E}{\partial S}\Big|_{J,V,\alpha}\right)\Big|_{S,V,\alpha} = \frac{\partial T}{\partial J}\Big|_{S,V,\alpha} = \frac{\{T, S, V, \alpha\}_{T,\Omega,P,\alpha}}{\{J, S, V, \alpha\}_{T,\Omega,P,\alpha}}, \quad (30)$$

or equivalently

$$\frac{\partial^2 E}{\partial S \partial J}\Big|_{V,\alpha} = \frac{\partial}{\partial S}\left(\frac{\partial E}{\partial J}\Big|_{S,V,\alpha}\right)\Big|_{J,V,\alpha} = \frac{\partial \Omega}{\partial S}\Big|_{J,V,\alpha} = \frac{\{\Omega, J, V, \alpha\}_{T,\Omega,P,\alpha}}{\{S, J, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (31)$$

What If...? Summary

- How to: Geometrization of Nonequilibrium thermodynamics and statistical physics?

Thank You!

- Partially supported by
  – The Bulgarian NSF Grants N28/5
  – The Sofia University Grant 80-10-33

Special gratitude to BPU11 Organizers!