Fundamental Incompatibility of the Yang-Mills Mass Gap and Asymptotic Freedom within Continuum Quantum Field Theory ver.2

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Abstract:
The existence of a mass gap in non-abelian Yang-Mills theory is a cornerstone prediction related to quark confinement, strongly supported by experimental observations and lattice simulations. The Clay Mathematics Institute designated its rigorous proof within continuum Quantum Field Theory (QFT) as a Millennium Prize Problem. Standard formulations rely on the Osterwalder-Schrader (OS) axioms to ensure a well-defined relativistic QFT possessing asymptotic freedom, the empirically verified weakening of interactions at high energies. This paper demonstrates a fundamental incompatibility between these established requirements. By analyzing the analytic structure of gauge-invariant two-point correlation functions via the Källén-Lehmann spectral representation (implied by OS axioms) constrained by the mass gap, and confronting it with the specific asymptotic behavior dictated by asymptotic freedom (derived from Renormalization Group analysis), a mathematical contradiction is rigorously derived. Specifically, the polynomial and logarithmic structure required by asymptotic freedom at high momentum cannot be reconciled with the asymptotic behavior allowed by the spectral representation for a theory with a mass gap and satisfying OS axioms. This incompatibility strongly suggests that the premise of a fundamental spacetime continuum, underlying standard QFT formulations, is inconsistent with the observed physical reality of the mass gap and asymptotic freedom.

Keywords: Yang-Mills Theory, Mass Gap, Asymptotic Freedom, Quantum Field Theory, Continuum Spacetime, Osterwalder-Schrader Axioms, Spectral Representation, Renormalization Group, Millennium Prize Problems, Axiomatic Field Theory.

Introduction

Quantum Chromodynamics (QCD), the theory describing the strong nuclear force binding quarks and gluons into hadrons, stands as a pillar of the Standard Model of particle physics (Fritzsch, Gell-Mann and Leutwyler 1973; Marciano and Pagels 1978). Based on the SU(3) non-abelian gauge group, as formulated in the seminal work on Yang-Mills theory (Yang and Mills 1954), QCD exhibits complex non-perturbative phenomena crucial for understanding the structure of matter. Foremost among these are confinement, the principle that colored quarks and gluons are not observed as free particles, and the existence of a mass gap, meaning that the spectrum of observable hadronic states is discrete and bounded below by the mass of the lightest hadron (pion, or glueball in pure Yang-Mills theory), denoted by $\Delta > 0$ (Jaffe and Witten 2000). This contrasts sharply with the massless nature of gluons in the classical Lagrangian and the small current masses of light quarks.

The physical reality of the mass gap is empirically undeniable, evidenced by the observed spectrum of massive hadrons and the absence of massless strongly interacting particles. Furthermore, numerical simulations using Lattice Gauge Theory (LGT), a non-perturbative regularization of QCD on a discrete spacetime lattice, provide overwhelming evidence for both confinement and the mass gap (Wilson 1974; Creutz 1980; Bali et al. 2000; Lucini and Teper 2004). LGT calculations successfully predict hadron masses and demonstrate the exponential decay of correlation functions characteristic of a gapped theory.

Despite this physical and numerical evidence, establishing the existence of quantum Yang-Mills theory as a mathematically rigorous Quantum Field Theory (QFT) on continuum spacetime R^4, and proving it possesses a mass gap, remains a major unsolved problem. Its fundamental importance led the Clay Mathematics Institute to designate it as one of the seven Millennium Prize Problems (Jaffe and Witten 2000). The challenge requires constructing a theory satisfying the rigorous axioms of QFT – typically the Wightman axioms in Minkowski space or the Osterwalder-Schrader (OS) axioms in the Euclidean formulation (Streater and Wightman 1964; Osterwalder and Schrader 1973, 1975) – and proving $\Delta > 0$ within that axiomatic framework.

A further complication, and a crucial aspect of physical reality, is that QCD must also exhibit asymptotic freedom (Gross and Wilczek 1973; Politzer 1973). This property, essential for explaining high-energy scattering data, dictates that the strong coupling constant vanishes logarithmically at short distances (high momenta). A complete continuum theory must therefore reconcile this perturbative ultraviolet (UV) behavior with the non-perturbative, gapped infrared (IR) physics.

The standard assumption within the physics community is that such a consistent continuum theory exists, and the difficulty in proving its properties lies in the lack of sufficiently powerful mathematical tools to handle non-perturbative QFT in four dimensions. This paper challenges this fundamental assumption. We propose that the requirements of axiomatic QFT (specifically the OS axioms), the existence of a mass gap $\Delta > 0$, and the property of asymptotic freedom are, in fact, mathematically incompatible when imposed simultaneously on a Yang-Mills theory formulated on a spacetime continuum.

This work aims to demonstrate this incompatibility by analyzing the rigorous consequences of the OS axioms (particularly reflection positivity leading to spectral representations) and the mass gap condition on the analytic structure of correlation functions, and showing that this structure conflicts with the specific asymptotic behavior mandated by asymptotic freedom and the Renormalization Group. If this incompatibility holds, it suggests that the Millennium Problem, as currently formulated within the continuum framework, may be ill-posed, and that the observed physical phenomena necessitate a departure from the assumption of a fundamental spacetime continuum. This paper presents the mathematical argument for this incompatibility and explores its profound implications for fundamental physics.

Literature Review

The Yang-Mills existence and mass gap problem sits at the confluence of particle physics phenomenology, quantum field theory, and mathematical physics. The literature underpinning this challenge is vast; this review highlights the key areas relevant to the problem's formulation and the premises of our argument.

Yang-Mills Theory and QCD

The foundational work by Yang and Mills (1954) introduced non-abelian gauge theories as a generalization of electromagnetism. The application of this framework to the strong interactions, leading to QCD with gauge group SU(3), was developed subsequently (Fritzsch, Gell-Mann and Leutwyler 1973). Standard textbook treatments provide comprehensive introductions to the classical and quantum aspects of Yang-Mills theory and QCD (e.g., Peskin and Schroeder 1995; Weinberg 1996; Schwartz 2014).

Asymptotic Freedom

The discovery of asymptotic freedom by Gross and Wilczek (1973) and Politzer (1973) was a landmark achievement, demonstrating that non-abelian gauge theories become weakly coupled at high energies. This property provided the theoretical basis for applying perturbative methods to high-energy strong interaction processes and was crucial for establishing QCD as the correct theory. The calculation of the beta function and the use of the Renormalization Group (RG) are central to understanding this UV behavior (Coleman 1985; Collins 1984; Zinn-Justin 2002). Experimental verification is extensive, primarily from deep inelastic scattering and jet production (Particle Data Group 2022).

Confinement and the Mass Gap

In contrast to the perturbative UV regime, the low-energy (IR) behavior of QCD is dominated by non-perturbative effects: confinement and the mass gap. While confinement lacks a rigorous proof, the concept is well-established phenomenologically and supported by theoretical ideas like the area law for Wilson loops (Wilson 1974). The mass gap, the statement that the lowest excitation above the vacuum has strictly positive mass, is intrinsically linked to confinement. The formal statement of the Millennium Problem by Jaffe and Witten (2000) precisely articulates the need for a rigorous proof of existence and the mass gap $\Delta > 0$ within an axiomatic framework.

Lattice Gauge Theory (LGT)

Introduced by Wilson (1974), LGT provides the primary tool for non-perturbative calculations in QCD. By discretizing spacetime, it allows for numerical simulations, most notably using Monte Carlo methods (Creutz 1980). LGT simulations have provided compelling quantitative evidence for confinement (via the static quark potential and Wilson loop behavior) and the existence of a mass gap (by calculating hadron masses and correlation function decay) (DeGrand and DeTar 2006; Gattringer and Lang 2010). Seminal works established the strong coupling expansion, which demonstrates confinement and a mass gap in the limit of large coupling (Wilson 1974; Kogut and Susskind 1975). However, rigorously proving that the continuum limit (lattice spacing $a \to 0$) exists, is non-trivial, and retains these properties remains an open problem, essentially equivalent to the Millennium Problem itself.

Axiomatic and Constructive Quantum Field Theory

The rigorous formulation of QFT relies on axiomatic frameworks. The Wightman axioms define QFT in Minkowski space (Streater and Wightman 1964), while the Osterwalder-Schrader (OS) axioms provide equivalent conditions for Euclidean Green's functions (Osterwalder and Schrader 1973, 1975). These axioms ensure fundamental physical principles like locality, causality, and positivity of energy (via the spectral condition). Reflection positivity in the OS framework is key, implying the Källén-Lehmann spectral representation for two-point functions, which relates correlation functions to the mass spectrum of the theory (Källén 1952; Lehmann 1954; Bogoliubov, Logunov and Todorov 1975). Constructive QFT aims to rigorously build interacting QFT models satisfying these axioms. While successful for certain super-renormalizable theories in lower dimensions (Glimm and Jaffe 1987), constructing 4D Yang-Mills theory remains beyond current methods. The Millennium Problem statement explicitly requires satisfying such an axiomatic framework (Jaffe and Witten 2000).

Spectral Representation and Asymptotic Behavior

The Källén-Lehmann spectral representation imposes strong constraints on the analytic properties and asymptotic behavior of correlation functions (Bogoliubov, Logunov and Todorov 1975; Weinberg 1995). Simultaneously, the Renormalization Group and Operator Product Expansion (OPE) dictate the asymptotic behavior at large momenta based on asymptotic freedom (Collins 1984; Zinn-Justin 2002). The interplay and potential conflict between the constraints imposed by the spectral representation (reflecting the mass spectrum and axiomatic structure) and the specific logarithmic scaling laws mandated by asymptotic freedom form the basis of the argument presented in this paper. While often assumed compatible, the rigorous consequences of their simultaneous imposition in the context of the OS axioms for non-abelian theories have not, to our knowledge, been fully explored to the point of demonstrating incompatibility.

Research Questions

This study aims to address the following fundamental questions regarding the mathematical consistency of continuum Quantum Yang-Mills theory:

1. Can the core requirements derived from empirical observation and theoretical consistency – specifically, the Osterwalder-Schrader axioms for a well-defined QFT, the existence of a non-zero mass gap ($\Delta > 0$) consistent with confinement, and the property of asymptotic freedom consistent with high-energy scattering – be simultaneously satisfied by a non-abelian Yang-Mills theory formulated on a continuum spacetime R^4?
2. Does a rigorous analysis of the analytic structure of gauge-invariant correlation functions, specifically the constraints imposed by the Källén-Lehmann spectral representation (reflecting the OS axioms and the mass gap) on their asymptotic behavior, reveal a mathematical conflict with the specific high-momentum scaling laws dictated by asymptotic freedom and the Renormalization Group?
3. If such an incompatibility is demonstrated, what are the implications for the validity of the continuum spacetime assumption underlying standard QFT formulations, particularly concerning the Yang-Mills Millennium Prize Problem?

Methodology

This study employs a theoretical and mathematical methodology based on established principles of axiomatic quantum field theory and the Renormalization Group. The core approach involves demonstrating a contradiction by confronting the rigorous consequences of different fundamental requirements imposed on the theory.

The key steps are as follows:

1. Axiomatic Foundation: We assume the existence of a Euclidean quantum Yang-Mills theory satisfying the Osterwalder-Schrader (OS) axioms on R^4. This provides the rigorous basis for the analysis, particularly leveraging the consequences of reflection positivity.
2. Spectral Representation: We utilize the Källén-Lehmann spectral representation (or its appropriate generalization for composite operators) for the two-point Schwinger function $\tilde{S}_2(p^2)$ of a suitably chosen gauge-invariant local operator, such as O(x) = :Tr(F_{μν}F^{μν})(x):. This representation expresses $\tilde{S}_2(p^2)$ in terms of an integral over a non-negative spectral density $\rho(m^2)$.

\tilde{S}_2(p^2) = P(p^2) + \int_{0}^\infty dm^2 \frac{\rho(m^2)}{p^2 + m^2} where $P(p^2)$ is a subtraction polynomial determined by renormalization.

3. Imposing the Mass Gap: The physical requirement of a mass gap $\Delta > 0$ is implemented by constraining the support of the spectral density: $\rho(m^2) = 0$ for $0 < m^2 < \Delta^2$. Any potential contribution at $m^2=0$ (related to vacuum expectation values) is absorbed into $P(p^2)$. The integral thus runs from Δ^2 to infinity.
$$\tilde{S}_2(p^2) = P(p^2) + \int_{\Delta^2}^\infty dm^2 \frac{\rho_c(m^2)}{p^2 + m^2} \quad (\rho_c \geq 0)$$

4. Imposing Asymptotic Freedom: The requirement of asymptotic freedom dictates the specific asymptotic behavior of $\tilde{S}_2(p^2)$ as $p^2 \to \infty$. This behavior is derived from Renormalization Group analysis and the Operator Product Expansion, incorporating the logarithmic decay of the running coupling constant $\alpha_s(p^2)$. For $O = :Tr(F^2):$, this behavior is expected to be of the form:
$$\tilde{S}_2(p^2) \sim C_0 \frac{(p^2)}{[\ln(p^2/\Lambda^2)]^k} \quad \text{as } p^2 \to \infty$$
for known constants C_0 and $k > 0$.

5. Asymptotic Analysis and Confrontation: We analyze the asymptotic behavior ($p^2 \to \infty$) of the spectral representation derived in Step 3. This involves determining the asymptotic form of the integral term and the constraints on the subtraction polynomial $P(p^2)$ from renormalization theory (based on the operator's dimension).

6. Derivation of Contradiction: We compare the asymptotic behavior derived from the spectral representation (Step 5) with the behavior required by asymptotic freedom (Step 4). The methodology aims to rigorously prove that these two asymptotic forms are mathematically irreconcilable for any choice of non-negative spectral density ρ_c satisfying the mass gap constraint and any allowed subtraction polynomial $P(p^2)$.

This method relies solely on the mathematical consequences of the foundational assumptions (OS axioms, mass gap, asymptotic freedom) and does not depend on specific models or approximations beyond standard, well-established QFT results regarding spectral representations, renormalization, and RG/OPE asymptotics. The goal is to expose a fundamental inconsistency within the continuum framework itself when subjected to these combined requirements.

Results: Proof of Incompatibility

In this section, we present the mathematical proof demonstrating the incompatibility between the Osterwalder-Schrader (OS) axioms, the existence of a mass gap ($\Delta > 0$), and the property of asymptotic freedom for continuum Yang-Mills theory on R^4. The proof follows the methodology outlined.

Theorem: No set of Euclidean Schwinger functions $\{S_n\}$ for a non-abelian Yang-Mills theory on R^4 can simultaneously satisfy the Osterwalder-Schrader axioms, exhibit a mass gap $\Delta > 0$, and possess the property of asymptotic freedom.

Proof:

Setup from OS Axioms and Mass Gap:

Assume such a set of Schwinger functions exists. Consider the two-point function $S_2(x-y) = \langle O(x)\, O(y)\rangle$ for the gauge-invariant, local operator $O(x) = :\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})(x):$, which has engineering dimension $d=4$. By the OS axioms (specifically reflection positivity) and the mass gap condition ($\Delta > 0$), its Fourier transform $\tilde{S}_2(p^2)$ admits the following spectral representation (Källén 1952; Lehmann 1954; Bogoliubov, Logunov and Todorov 1975):

$$\tilde{S}_2(p^2) = P(p^2) + \int_{\Delta^2}^\infty dm^2 \frac{\rho_c(m^2)}{p^2 + m^2} \quad \quad \text{(Eq. 1)}$$

Here, p^2 is the squared Euclidean momentum. The spectral density $\rho_c(m^2)$ is non-negative ($\rho_c(m^2) \geq 0$) and vanishes for $m^2 < \Delta^2$ due to the mass gap. $P(p^2)$ is a subtraction polynomial required for renormalization of the composite operator $O(x)$.

Constraint on Subtraction Polynomial:

Standard renormalization theory (power counting, BPHZ theorem, or Epstein-Glaser approach) dictates the maximum degree of the subtraction polynomial based on the superficial degree of divergence of the correlation function (Collins 1984; Zinn-Justin 2002). For the two-point function of an operator with dimension $d=4$, the superficial degree of divergence is typically calculated to be 0 (logarithmic). This restricts the subtraction polynomial $P(p^2)$ to be at most linear in p^2 to absorb divergences and implement renormalization conditions.

$$P(p^2) = a_0 + a_1 p^2 \quad \quad \text{(Eq. 2)}$$

where a_0 and a_1 are renormalization constants.

Asymptotic Behavior from Spectral Representation:

We analyze the behavior of $\tilde{S}_2(p^2)$ from Eq. 1 as $p^2 \to \infty$. The integral term $I(p^2) = \int_{\Delta^2}^\infty dm^2\, \rho_c(m^2) / (p^2 + m^2)$ can be expanded for large p^2:

$$I(p^2) = \frac{1}{p^2} \int_{\Delta^2}^\infty dm^2 \rho_c(m^2) - \frac{1}{(p^2)^2} \int_{\Delta^2}^\infty dm^2 m^2 \rho_c(m^2) + O\left(\frac{1}{(p^2)^3}\right) \quad \quad (Eq. 3)$$

This assumes the convergence of the first few moments of $\rho\_c(m^2)$, which is necessary for the temperedness (OS0 axiom) of the distribution $\tilde{S}_2(p^2)$. The crucial point is that $I(p^2) = O(1/p^2)$, meaning it vanishes at least as fast as $1/p^2$ for large p^2. Substituting Eq. 2 and the behavior from Eq. 3 into Eq. 1, the asymptotic behavior derived from the OS axioms and the mass gap is:

$$\tilde{S}_2(p^2) \sim a_1 p^2 + a_0 + O(1/p^2) \quad \text{as } p^2 \to \infty \quad \quad (Eq. 4)$$

Asymptotic Behavior from Asymptotic Freedom:

Asymptotic freedom and the Renormalization Group dictate the true UV asymptotic behavior of $\tilde{S}_2(p^2)$ (Gross and Wilczek 1973; Politzer 1973; Collins 1984). For the operator $O = :Tr(F^2):$, the OPE and RG analysis predict:

$$\tilde{S}_2(p^2) \sim C_0 \frac{(p^2)}{[\ln(p^2/\Lambda^2)]^k} \quad \text{as } p^2 \to \infty \quad \quad (Eq. 5)$$

Here, C_0 is a non-zero constant calculable in perturbation theory, Λ is the intrinsic scale of the theory (e.g., Λ_QCD), and k is a positive power related to the beta function and anomalous dimensions (the exact value is not crucial, only that $k > 0$). This behavior reflects the logarithmic vanishing of the effective coupling at high momentum.

Contradiction:

We now compare the asymptotic behavior derived from the spectral representation (Eq. 4) with that required by asymptotic freedom (Eq. 5).
Comparing the leading terms as $p^2 \to \infty$:

$$a_1 p^2 \quad \text{vs} \quad C_0 \frac{p^2}{[\ln(p^2/\Lambda^2)]^k}$$

Since $1 / [\ln(p^2/\Lambda^2)]^k \to 0$ as $p^2 \to \infty$, for these two forms to be asymptotically equivalent, the coefficient a_1 must be zero ($a_1 = 0$).
With $a_1 = 0$, the behavior from the spectral representation (Eq. 4) becomes:

$$\tilde{S}_2(p^2) \sim a_0 + O(1/p^2)$$

This means $\tilde{S}_2(p^2)$ approaches a constant a_0 (or vanishes if $a_0=0$) as $p^2 \to \infty$.

However, the behavior required by asymptotic freedom (Eq. 5) is:

$$\tilde{S}_2(p^2) \sim C_0 \frac{p^2}{[\ln(p^2/\Lambda^2)]^k}$$

Since $C_0 \neq 0$ and $k > 0$, this term grows unboundedly as $p^2 \to \infty$ (albeit slower than linearly due to the logarithm).

A function approaching a constant (a_0) cannot be asymptotically equivalent to a function that grows unboundedly ($\sim p^2 / (\ln p^2)^k$). This is a direct mathematical contradiction.

Conclusion of Proof:

The assumption that a set of Schwinger functions exists satisfying simultaneously the OS axioms, the mass gap condition ($\Delta > 0$), and asymptotic freedom leads to an unavoidable mathematical contradiction when analyzing the asymptotic behavior of the two-point function $\langle :\mathrm{Tr}(F^2)(x): \, :\mathrm{Tr}(F^2)(0):\rangle$. Therefore, such a theory formulated on continuum spacetime R^4 cannot exist.

Discussion

The mathematical proof presented demonstrates a fundamental incompatibility between three core tenets assumed to hold for Quantum Yang-Mills theory within the standard continuum Quantum Field Theory (QFT) framework: the Osterwalder-Schrader (OS) axioms ensuring a well-defined relativistic quantum theory, the existence of a mass gap ($\Delta > 0$) as required by experimental observation and confinement, and the property of asymptotic freedom governing the theory's high-energy behavior. The contradiction arises from the irreconcilable constraints these tenets impose on the asymptotic behavior of gauge-invariant correlation functions, specifically the two-point function of the operator $:\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}):$.

The Källén-Lehmann spectral representation, a rigorous consequence of the OS axioms (particularly reflection positivity) and Lorentz invariance, combined with the mass gap condition, dictates that the momentum-space correlator $\tilde{S}_2(p^2)$ can only grow polynomially at large p^2 (determined by renormalization subtractions) plus terms that vanish as $p^2 \to \infty$. Standard renormalization theory for this dimension-4 operator restricts the polynomial growth to be at most linear ($\sim a_1 p^2$). In contrast, asymptotic freedom, derived from Renormalization Group analysis, mandates a specific growth proportional to p^2 but suppressed by powers of $\ln(p^2)$. Our analysis showed that these two asymptotic forms cannot be matched for any allowed spectral density and subtraction scheme. The requirement $a_1 = 0$ needed to match the logarithmic suppression leads to an overall decay or constant behavior from the spectral representation, contradicting the required growth from asymptotic freedom.

This result has profound implications. The Yang-Mills Millennium Prize Problem explicitly asks for a construction satisfying the axioms and exhibiting a mass gap (Jaffe and Witten 2000). Asymptotic freedom, while not always stated as an explicit requirement in the mathematical formulation, is an undeniable physical property of the theory governing the strong interactions (QCD) and is implicitly assumed in any physically relevant formulation. Our finding suggests that the problem, as posed within the framework of continuum QFT on R^4, may not have a solution. The failure to construct such a theory might not stem from insufficient mathematical techniques but from a fundamental inconsistency in the requirements imposed on the continuum structure.

The standard view holds that the continuum limit of Lattice Gauge Theory (LGT) should yield the desired theory. LGT simulations provide strong evidence for the mass gap and can incorporate asymptotic freedom through the running of the bare coupling with the lattice spacing a. However, the rigorous proof that the a → 0 limit exists and satisfies *all* OS axioms while retaining *both* the mass gap and the correct *continuum* asymptotic freedom scaling (with its specific logarithmic structure) is precisely what is lacking. Our result suggests that such a limit, if it were to satisfy the OS axioms, cannot simultaneously possess the other two properties. This might indicate that LGT, while an excellent computational tool and regularization, does not converge to a continuum QFT satisfying all physical requirements, or that the nature of the continuum limit is more subtle than typically assumed.

The demonstrated incompatibility points towards the foundational assumption of a spacetime continuum as the source of the conflict. The rigidity of analytic structures and scaling laws inherent in continuum QFT appears unable to accommodate the distinct physical behaviors observed empirically in the infrared (mass gap) and the ultraviolet (asymptotic freedom) within a single, self-consistent axiomatic framework. This echoes historical paradigm shifts where phenomena inexplicable within an existing framework necessitated a fundamental change in underlying concepts – such as the transition from classical mechanics to quantum mechanics to explain atomic phenomena.

If the continuum is not the fundamental structure, alternative descriptions of spacetime become necessary. Frameworks based on discrete spacetime (like LGT, but perhaps viewed as fundamental rather than a regulator), causal sets, loop quantum gravity (which features a discrete quantum geometry), or novel approaches like the Simplicial Discrete Informational Spacetime (SDIS) framework proposed by the author (Karazoupis 2025) may offer paths forward. Such frameworks might possess different analytic structures or inherent scales that could naturally accommodate both a mass gap and asymptotic freedom without the contradictions encountered in the continuum. For instance, the analysis within the SDIS framework at strong coupling already indicates a natural emergence of the mass gap (Karazoupis 2025). Exploring whether such frameworks can rigorously incorporate asymptotic freedom while preserving the gap becomes a crucial direction.

In summary, the mathematical contradiction derived herein between the OS axioms, mass gap, and asymptotic freedom challenges the foundational assumption of continuum spacetime for Yang-Mills theory. It reframes the Millennium Problem, suggesting the core difficulty lies not merely in mathematical complexity but in the limitations of the continuum paradigm itself when confronted with the full spectrum of physical reality observed for the strong interactions.

Conclusion

This paper has investigated the mathematical consistency of incorporating the fundamental physical requirements of non-abelian Yang-Mills theory within the standard framework of continuum Quantum Field Theory (QFT) on R^4, as governed by the Osterwalder-Schrader (OS) axioms. The core requirements considered were the existence of a mass gap ($\Delta > 0$), reflecting confinement and the observed hadron spectrum, and the property of asymptotic freedom, describing the weakening of interactions at high energies.

By analyzing the rigorous consequences of the OS axioms and the mass gap condition on the spectral representation of the two-point correlation function for the gauge-invariant operator :Tr(F_{μν}F^{μν}):, and comparing the resulting asymptotic behavior with that mandated by asymptotic freedom and the Renormalization Group, we have demonstrated a fundamental mathematical incompatibility. The analytic structure allowed by the spectral representation for a gapped theory cannot reproduce the specific logarithmic dependence required by asymptotic freedom at high momentum within the constraints imposed by renormalization and the OS axioms.

The central result of this work is the conclusion that continuum Yang-Mills theory cannot simultaneously satisfy the OS axioms, possess a mass gap $\Delta > 0$, and exhibit asymptotic freedom. This finding challenges the standard assumption underlying the Yang-Mills Millennium Prize Problem – namely, that such a consistent continuum theory exists and awaits rigorous construction. Our analysis suggests that the persistent difficulty in solving this problem may not solely be technical, but may originate from an inherent inconsistency within the continuum QFT framework itself when confronted with the full range of empirical data for the strong interactions.

The incompatibility implies that the foundational assumption of a smooth spacetime continuum may be an approximation that breaks down at the level required to fully describe quantum Yang-Mills theory. Just as classical physics yielded to quantum mechanics, the continuum description may need to yield to a more fundamental, potentially non-continuous, description of spacetime to resolve this conflict. This motivates further investigation into alternative frameworks, such as those based on discrete spacetime structures, quantum geometry, or other novel paradigms that might naturally reconcile the observed infrared (mass gap) and ultraviolet (asymptotic freedom) behaviors without contradiction.

In conclusion, the empirical realities of the mass gap and asymptotic freedom in Yang-Mills theory, when interpreted through the lens of rigorous axiomatic QFT, appear to provide compelling mathematical evidence against the fundamental nature of continuum spacetime.

References

Bali, G.S., et al. (UKQCD Collaboration) (2000) 'Static potentials and glueball masses from QCD simulations with Wilson sea quarks'. *Physical Review D*, 62(5), 054503. doi: 10.1103/PhysRevD.62.054503.

Bogoliubov, N.N., Logunov, A.A. and Todorov, I.T. (1975) *Introduction to Axiomatic Quantum Field Theory*. Reading, MA: W. A. Benjamin.

Coleman, S. (1985) *Aspects of Symmetry*. Cambridge: Cambridge University Press.

Collins, J.C. (1984) *Renormalization*. Cambridge: Cambridge University Press.

Creutz, M. (1980) ‘Monte Carlo study of quantized SU(2) gauge theory’. *Physical Review D*, 21(8), pp. 2308–2315. doi: 10.1103/PhysRevD.21.2308.

DeGrand, T. and DeTar, C. (2006) *Lattice Methods for Quantum Chromodynamics*. Singapore: World Scientific.

Fritzsch, H., Gell-Mann, M. and Leutwyler, H. (1973) ‘Advantages of the color octet gluon picture’. *Physics Letters B*, 47(4), pp. 365–368. doi: 10.1016/0370-2693(73)90625-4.

Gattringer, C. and Lang, C.B. (2010) *Quantum Chromodynamics on the Lattice: An Introductory Presentation*. Berlin, Heidelberg: Springer Berlin Heidelberg. doi: 10.1007/978-3-642-01850-3.

Glimm, J. and Jaffe, A. (1987) *Quantum Physics: A Functional Integral Point of View*. 2nd edn. New York: Springer-Verlag.

Gross, D.J. and Wilczek, F. (1973) ‘Ultraviolet Behavior of Non-Abelian Gauge Theories’. *Physical Review Letters*, 30(26), pp. 1343–1346. doi: 10.1103/PhysRevLett.30.1343.

Jaffe, A. and Witten, E. (2000) ‘Quantum Yang-Mills Theory’. In: *The Millennium Prize Problems*. Cambridge, MA: Clay Mathematics Institute, pp. 129–152. Available at: http://www.claymath.org/sites/default/files/yangmills.pdf

Källén, G. (1952) ‘On the definition of the Renormalization Constants in Quantum Electrodynamics’. *Helvetica Physica Acta*, 25, pp. 417–434.

Karazoupis, M. (2025) *Complete Theory of Simplicial Discrete Informational Spacetime: Towards a Predictive and Testable Theory of Quantum Spacetime*. [Preprint]. doi: 10.20944/preprints202504.0029.v1.

Karazoupis, M. (2025) *Existence of a Mass Gap in SU(3) Yang-Mills Theory within the Simplicial Discrete Informational Spacetime Framework: A Strong Coupling Analysis*. [Preprint]. doi: 10.20944/preprints202504.1265.v1.

Kogut, J. and Susskind, L. (1975) 'Hamiltonian formulation of Wilson's lattice gauge theories'. *Physical Review D*, 11(2), pp. 395–408. doi: 10.1103/PhysRevD.11.395.

Lehmann, H. (1954) ‘Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder’. *Il Nuovo Cimento*, 11(4), pp. 342–357. doi: 10.1007/BF02783624.

Lucini, B. and Teper, M. (2004) 'SU(N) gauge theories in four dimensions: exploring the approach to N = ∞'. *Journal of High Energy Physics*, 2004(06), 050. doi: 10.1088/1126-6708/2004/06/050.

Marciano, W. and Pagels, H. (1978) ‘Quantum chromodynamics’. *Physics Reports*, 36(3), pp. 137–276. doi: 10.1016/0370-1573(78)90174-1.

Osterwalder, K. and Schrader, R. (1973) ‘Axioms for Euclidean Green’s functions’. *Communications in Mathematical Physics*, 31(2), pp. 83–112. doi: 10.1007/BF01645738.

Osterwalder, K. and Schrader, R. (1975) ‘Axioms for Euclidean Green’s functions. II’. *Communications in Mathematical Physics*, 42(3), pp. 281–305. doi: 10.1007/BF01608978.

Particle Data Group (Workman, R.L. et al.) (2022) ‘Review of Particle Physics’. *Progress of Theoretical and Experimental Physics*, 2022(8), 083C01. doi: 10.1093/ptep/ptac097.

Peskin, M.E. and Schroeder, D.V. (1995) *An Introduction to Quantum Field Theory*. Reading, MA: Addison-Wesley Publishing Company.

Politzer, H.D. (1973) ‘Reliable Perturbative Results for Strong Interactions?’. *Physical Review Letters*, 30(26), pp. 1346–1349. doi: 10.1103/PhysRevLett.30.1346.

Schwartz, M.D. (2014) *Quantum Field Theory and the Standard Model*. Cambridge: Cambridge University Press.

Streater, R.F. and Wightman, A.S. (1964) *PCT, Spin and Statistics, and All That*. New York: W. A. Benjamin.

Weinberg, S. (1995) *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge: Cambridge University Press.

Weinberg, S. (1996) *The Quantum Theory of Fields, Volume 2: Modern Applications*. Cambridge: Cambridge University Press.

Wilson, K.G. (1974) ‘Confinement of quarks’. *Physical Review D*, 10(8), pp. 2445–2459. doi: 10.1103/PhysRevD.10.2445.

Yang, C.N. and Mills, R.L. (1954) ‘Conservation of Isotopic Spin and Isotopic Gauge Invariance’. *Physical Review*, 96(1), pp. 191–195. doi: 10.1103/PhysRev.96.191.

Zinn-Justin, J. (2002) *Quantum Field Theory and Critical Phenomena*. 4th edn. Oxford: Oxford University Press.