The contribution Hans Adolph Buchdahl *f(R)* gravity

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Abstract. Hans Adolph Buchdahl was born in Mainz in German in a Jewish family. He was Australian Physicist and he was born 7 July 1919 and died 7 January 2010. He worked on general relativity, thermodynamics and optics. He was the founder of the modified theory of gravity of the *f(R)* type, where *f(R)* is general functions of Ricci scalar, unlike Einstain's general theory of relativity *f(R)=R*. He proposed *f(R)* gravity first in in his paper 1970 [1]. He set up the field equations of *f(R)* gravity. Also he is known for developing Buchdahl's theorem which is an relation between the mass and radius in the static, spherically symmetric matter configurations.
This paper will present his historical work on the creation of *f(R)* gravity as the founder of *f(R)* gravity, and also in one chapter it will be represented Buchdahl's theorem

Hans Adolf Buchdahl was born in Mainz, Germany, in a Jewish family. His older brother Gerd Buchdahl was a philosopher in science. In 1933, Gerd and Hans went to England, to escape the Nazi government. At London, he finished a BSc and became the Associate of the Royal College of Science (ARCS) from Imperial College. When World War II began, the UK government all German nationals including many Jewish refugees deported to Australia. In July 1940, Hans came to Australia together with Gerd on board the HMT Dunera. First he was initially at Hay in New South Wales, then at the Tatura centre in Victoria in May 1941. He had mathematical abilities so he was released on a guarantor program and got the job at the Physics Department of the University of Tasmania in Hobart. There he had to assist the overloaded teaching staff involved in wartime military research in optics. In 1949, he received his doctorate from University of Tasmania. In 1956, he was awarded a PhD from Imperial College London. From 1963 he was professor and head of the Department of Theoretical Physics in the Faculty of Science at the Australian National University in Canberra until he became pensioner 1984–1985. He married Pamela Wann in 1950 and they had three children. He died in Adelaide, Australia, on 7 January 2010.

BUCHDAHL THEOREM

Buchdahl proposed spherically symmetric general time invariant metric which take form $ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\varphi^2$, where ν and λ are functions of radius *r* and *c* is velocity of light. Matric elements of metric tensor are:

$g_{00} = e^{\nu(r)}, g_{11} = -e^{\lambda(r)}, g_{22} = -r^2, g_{33} = -r^2sin^2\theta,$

Cristoffel symbols are given: $\Gamma^{\alpha}_{\varepsilon\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\varepsilon,\nu} + g_{\sigma\nu,\varepsilon} - g_{\varepsilon\nu,\sigma})$, where $\Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{\partial\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}}$ and $g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$. Cristoffel symbols $\Gamma^{\alpha}_{\varepsilon\nu}$ different from zero for Buchdahl proposed metric are: $\Gamma^1_{00} = \frac{e^{\nu(r)-\lambda(r)}}{2}\frac{d\nu(r)}{dr}, \Gamma^1_{11} = \frac{d\lambda(r)}{2dr}, \Gamma^1_{22} = -re^{-\lambda(r)}$

$\Gamma^1_{33} = -re^{-\lambda(r)}sin^2\theta\ ,\ \Gamma^1_{10} = \Gamma^1_{01} = \frac{d\nu}{2dr}$

$\Gamma^2_{21} = \Gamma^3_{31} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^2_{33} = -sin\theta cos\theta, \Gamma^3_{23} = \Gamma^3_{32} = ctg\theta.$

Ricci tensor $R_{\mu\nu}$ and Ricci scalar *R* are expressed:

$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\varepsilon\nu}\Gamma^{\varepsilon}_{\mu\alpha} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\alpha}$

$R_{00} = e^{\nu(r)-\lambda(r)}\left(-\frac{d^2\nu}{2dr^2} - \frac{d\nu}{rdr} + \frac{d\nu}{4dr}\frac{d\lambda}{dr} - \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2\right), R_{33} = R_{22}sin^2\theta$

$$R_{11} = \frac{d^2\nu}{2dr^2} - \frac{d\lambda}{rdr} - \frac{d\nu}{4dr}\frac{d\lambda}{dr} + \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2, R_{22} = -1 + e^{-\lambda}\left(\frac{r}{2}\left(\frac{d\nu}{dr} - \frac{d\lambda}{dr}\right) + 1\right),$$

$$\mathrm{R} = e^{-\lambda(r)}\left(-\frac{d^2\nu}{dr^2} - \frac{2d\nu}{rdr} + \frac{d\nu}{2dr}\frac{d\lambda}{dr} - \frac{1}{2}\left(\frac{d\nu}{dr}\right)^2 + \frac{2d\lambda}{rdr} - \frac{2}{r^2}\right) + \frac{2}{r^2},$$

Einstein equation is: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \pm\frac{8\pi G}{c^4}T_{\mu\nu}$

To demand continuity of the stress-energy tensor:

$\nabla_{\mu}T^{\mu}_{\nu} = 0$, where $T_{00} = \rho c^2 e^{\nu}, T_{11} = T_{22} = T_{33} = Pe^{\lambda}$

There are three equations of field

$$\frac{8\pi G}{c^2}\rho r^2 = 1 - e^{-\lambda} + r\frac{d\lambda}{dr}e^{-\lambda} \ \rightarrow\ e^{-\lambda} = 1 - \frac{2GM}{rc^2}$$

$$-\frac{8\pi G}{c^4}Pr^2 = 1 + e^{-\lambda}\left(-r\frac{d\nu}{dr} - 1\right)$$

Second equation get next shape: $\frac{d\nu}{dr} = \frac{1}{r}\left[1 - \frac{2GM}{rc^2}\right]^{-1}\left[\frac{2GM}{rc^2} + \frac{8\pi G}{c^4}Pr^2\right]$.

The relation of third equation $\nabla_{\mu}T^{\mu}_{\nu} = 0$ is: $\frac{dP}{dr} = -\left(\frac{\rho c^2+P}{2}\right)\frac{d\nu}{dr}$.

So the final equation is Tolman-Oppenheimer-Volkoff equation:

$$\frac{dP}{dr} = -\frac{1}{r}\left(\frac{\rho c^2 + P}{2}\right)\left[1 - \frac{2GM}{rc^2}\right]^{-1}\left[\frac{2GM}{rc^2} + \frac{8\pi G}{c^4}Pr^2\right]$$

A simple model: the density is constant out to the surface of the star:

$$\rho(r) = \begin{Bmatrix}\rho & r < R\\ 0 & r > R\end{Bmatrix}, m(r) = \begin{Bmatrix}\frac{4}{3}\pi r^3\rho & r < R\\ M = \frac{4}{3}\pi R^3\rho & r > R\end{Bmatrix}$$

The expression for the pressure in this simple model is:

$$P(r) = \rho c^2\frac{R\sqrt{R - \frac{2GM}{c^2}} - \sqrt{R^3 - \frac{2GM}{c^2}r^2}}{\sqrt{R^3 - \frac{2GM}{c^2}r^2} - 3R\sqrt{R - \frac{2GM}{c^2}}}$$

The pressure and density must be greater than zero and *D(r)* becomes less than zero for $r = 0$.

$$D(r) = \sqrt{R^3 - \frac{2GM}{c^2}r^2} - 3R\sqrt{R - \frac{2GM}{c^2}}\ ,\quad D(0) = \sqrt{R^3} - 3R\sqrt{R - \frac{2GM}{c^2}} < 0$$

$$M < \frac{4c^2R}{9G}\ , R > \frac{9GM}{4c^2} = \frac{9}{8}Rs$$

[1] H. A. Buchdahl, General Relativistic Fluid Spheres,Physical Review Vol. 116, No 4, (1959) 1025.
[2] Sean Carroll, An introduction to general relativity SPACETIME AND GEOMETRY, ISBN 0-8953-8732-3, Addison Wesley, New York.
[3] H.A. Buchdahl, Non-linear langragians and cosmological theory, MNRAS, 150, (1970), 1-8.

$f(R)$ GRAVITY

Theory of general relativity is based on Einstain's equations which calculate from a variation of the density field Langragian $\Lambda = R + constant$ (R is Ricci scalar) and equations have the next shape: $P_{\mu\nu} = \pm kT_{\mu\nu}$, where k is constant, $P_{\mu\nu}$ is geometric tensor which depends on metric tensor $g_{\mu\nu}$ and their derivates, and $T_{\mu\nu}$ is tensor of energy momentum which presents the source of the gravitation field. Also tensor $T_{\mu\nu}$ satisfies

$$\nabla^{\mu}T_{\mu\nu} = 0$$

where ∇^{μ} is covariant derivate. In Einstain theory of general relativity $P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu}$. Buchdahl proposed to put the density Langragian is $\Lambda = f(R)$, general function of Ricci scalar R. He proposed following expressions for function $f(R) = \sum_0^n f_nR^n$ where f_n is constant which dimension is $(length)^{2n-2}$ and more realistic function $f(R) = \sqrt{\left(\left(R + \frac{1}{2l^2}\right)^2 + \frac{1}{2l^4}\right)}$, where *l* is fundamental length.

DERIVATION OF FIELD EQUATION FROM LANGRAGIAN $L = \sqrt{-g}f(R)$

Determinant of metric tensor is g. Next equations are useful expressions. Variation of

$$\delta[\sqrt{-g}f] = \sqrt{-g}\left(\frac{df}{dR}\delta R + \frac{1}{2}g^{\alpha\beta}f\delta g_{\alpha\beta}\right)$$

$$= \sqrt{-g}\left(\frac{df}{dR}g^{\alpha\beta}\delta R_{\alpha\beta} + (-\frac{1}{2}g_{\alpha\beta}f + \frac{df}{dR}R_{\alpha\beta})\delta g^{\alpha\beta}\right)$$

$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\varepsilon\nu}\Gamma^{\varepsilon}_{\mu\alpha} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\alpha}$

$\delta R_{\mu\nu} = \delta\Gamma^{\alpha}_{\mu\alpha,\nu} - \delta\Gamma^{\alpha}_{\mu\nu,\alpha}$

$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\nabla_{\nu}\delta g_{\beta\mu} + \nabla_{\mu}\delta g_{\beta\nu} - \nabla_{\beta}\delta g_{\mu\nu})$

$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\sigma}(g^{\mu\sigma}\delta\Gamma^{\alpha}_{\mu\alpha} - g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu})$

$g^{\mu\nu}\delta R_{\mu\nu} = -g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\delta g^{\mu\nu} + \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu}$

$$\delta(\sqrt{-g}f) = \sqrt{-g}\frac{df}{dR}(-g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\delta g^{\mu\nu} + \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu}) + \sqrt{-g}(-\frac{1}{2}g_{\mu\nu}f + \frac{df}{dR}R_{\mu\nu})\delta g^{\mu\nu}$$

$$P_{\mu\nu} = -g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\frac{df}{dR} + \nabla_{\mu}\nabla_{\nu}\frac{df}{dR} - \frac{1}{2}g_{\mu\nu}f + \frac{df}{dR}R_{\mu\nu}$$

And the equation from Buchdahl paper is:

$$\boldsymbol{g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\frac{df}{dR} - \nabla_{\mu}\nabla_{\nu}\frac{df}{dR} + \frac{1}{2}g_{\mu\nu}f - \frac{df}{dR}R_{\mu\nu} = kT_{\mu\nu}}$$

$$\nabla_{\mu}\nabla_{\nu}\frac{df}{dR} = \frac{d^2f}{dR^2}\nabla_{\mu}\nabla_{\nu}R + \frac{d^3f}{dR^3}\nabla_{\mu}R\nabla_{\nu}R$$

$$-\frac{d^2f}{dR^2}\nabla_{\mu}\nabla_{\nu}R - \frac{d^3f}{dR^3}\nabla_{\mu}R\nabla_{\nu}R + g_{\mu\nu}\frac{d^3f}{dR^3}\nabla_{\alpha}\mathrm{R}\nabla^{\alpha}R + g_{\mu\nu}\frac{d^2f}{dR^2}\nabla_{\alpha}\nabla^{\alpha}R + \frac{1}{2}g_{\mu\nu}f - \frac{df}{dR}R_{\mu\nu} = kT_{\mu\nu}$$

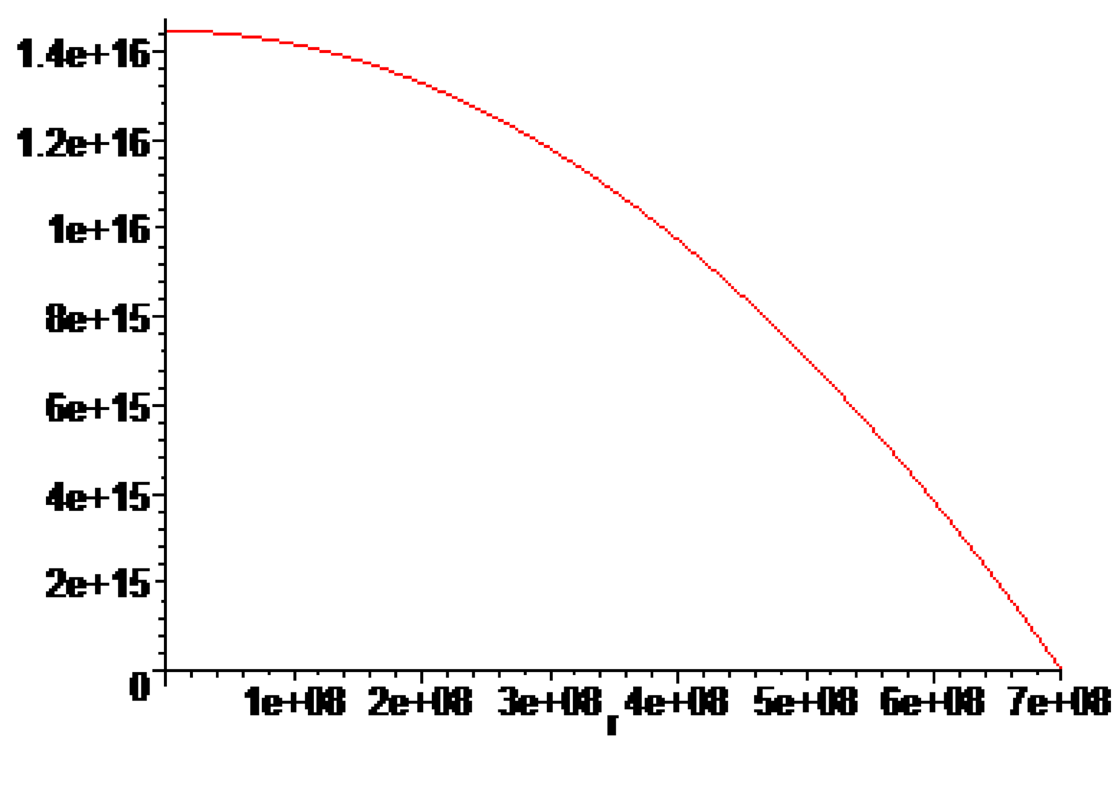
Final equation in Buchdahl paper from 1970 year is:

$$\boldsymbol{(-\nabla_{\mu}R\nabla_{\nu}R + g_{\mu\nu}\nabla_{\alpha}R\nabla^{\alpha}R)\frac{d^3f}{dR^3} + (-\nabla_{\mu}\nabla_{\nu}R + g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R)\frac{d^2f}{dR^2} - \frac{df}{dR}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f = kT_{\mu\nu}}$$

∇_{μ} is covariant derivate. For scalar $\nabla_{\mu} = \partial_{\mu}$

Conclusion for Buchdahl theorem: Central pressure of star Sun in function of radius. This central pressure diverges at the Buchdahl bound.
The value R>9GM/4c^2 is an absolute lower limit for all static fluid spheres whose density does not increase outwards.

Spherical static symmetry.
Termodynamic stability: density of matter is a non increasing function of radial distance. Mass to radius ratio limit is 4/9.
If an object exceeds the Buchdahl limit it means that exist new type object which not covered by classical theory. The central pressure p(0) will be infinity, if the mass exceeds Buchdahl limit.



Central pressure P(0.01Pa) of star Sun in function of radius r(m). R=700000km, ρ=1500kg/ m^3